Lecture 1
Introduction

1 The Laplace operator

On the 3-dimensional Euclidean space, the Laplace operator (or Laplacian) is the linear differential operator

$$\Delta : \begin{cases} C^2(\mathbb{R}^3) \to C^0(\mathbb{R}^3) \\ f \mapsto \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}, \end{cases}$$

where $C^k(\mathbb{R}^3)$, for any non-negative integer k, denotes the set of all the functions on $\mathbb{R}^3$ whose partial derivatives of order less than or equal to k exist and are continuous, and $\partial^2 f/\partial x_j^2$, for $j = 1, 2, 3$, denotes the second derivative of f with respect to the jth variable.

It is simple to generalize the Laplace operator to arbitrary domains in n-dimensional Euclidean spaces, for any natural number n. Let Ω be an open subset of $\mathbb{R}^n$. The Laplace operator on Ω is the operator

$$\Delta : \begin{cases} C^2(\Omega) \to C^0(\Omega) \\ f \mapsto \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}, \end{cases}$$

where $C^k(\Omega)$ and $\partial^2 f/\partial x_j^2$ are defined similarly as above. More generally, as we will see later, the Laplace operator can be generalized to any Riemannian manifold (M, g):

$$\Delta_g : \begin{cases} C^2(M) \to C^0(M) \\ f \mapsto \Delta_g f. \end{cases}$$

This generalization of the Laplacian to Riemannian manifolds is called the *Laplace-Beltrami operator*. The study of this operator and the relations of its spectrum (the eigenvalues) to the geometry of the manifold (the dimension, volume, curvature, etc.) is called *spectral geometry*. Our goal in this text is to provide an introduction to this subject.

Remark 1. When we write “the Laplacian on Ω”, we mean “the Laplacian acting on functions defined on Ω”. As is common for linear operators, we usually write Δf instead of $\Delta(f)$.

2 Why the Laplacian?

One reason why the Laplacian is so important is that it appears in many equations of physics, mathematics, and applied sciences. For instance, the Laplace operator appears in the following equations:

- Laplace's equation:
$$\Delta u = 0.$$
- Poisson's equation:
$$-\Delta u = f.$$
- Helmholt's equation:
$$-\Delta u = \lambda u.$$
- Heat equation:
$$\frac{\partial u}{\partial t}(x,t) = \Delta_x u(x,t).$$
- Wave equation:
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \Delta_x u(x,t).$$
- Schrödinger's equation:
$$i\frac{\partial \psi}{\partial t}(x,t) = -\Delta_x \psi(x,t) + V(x)\psi(x,t).$$

Remark 2. The notation Δ_x indicates that the Laplacian is calculated with respect to the variable $x = (x_1, \ldots, x_n)$.

The above equations are important partial differential equations which are used in many contexts. For instance, Laplace's equation and Poisson's equation appear in gravitational theory and electrostatics, the heat equation and the wave equation are used in the study of diffusion processes and wave phenomena, and the Schrödinger equation is the fundamental equation of non-relativistic quantum mechanics. Thus the Laplacian is important because it appears in many important equations (this is a good enough reason to study the Laplacian). But we may proceed and ask the following question: Why all these basic equations, which are used in different contexts, contain the Laplace operator instead of some other differential operator?

The answer to the above question starts with the following observation. Consider the above equations on 3-dimensional Euclidean space. Although these basic equations appear in different contexts, they all have one thing in common: They are equations defined on Euclidean space which are supposed to describe some physical quantity (the unknown). Thus, the equations should agree with the general principles of physics. In particular, they should agree with the principle that asserts that the laws of physics are independent of position and direction. This implies that the spatial components of the equations should be invariant by translations and rotations in Euclidean space. In particular, this implies that the differential operators should be invariant by translations and rotations. This gives a clue to the answer, because the Laplacian is essentially the only (scalar) second-order linear differential operator that is invariant by translations and rotations:

Theorem 1 (Informal)**.** *Suppose that L is a second-order linear differential operator on $\mathbb{R}^n$, and suppose that T is a translation on $\mathbb{R}^n$ or a rotation on $\mathbb{R}^n$. Then L is invariant by T, that is,*

$$(Lf) \circ T = L(f \circ T) \quad \textit{for all } f \in \mathrm{Domain}(L),$$

if and only if

$$L = c_1 \Delta + c_0,$$

where c_0 and c_1 are constants.

In some sense, this explains why the Laplacian is important for the description of phenomena in $\mathbb{R}^n$. More generally, as we will see later, the Laplace-Beltrami operator on a Riemannian manifold (M, g) is invariant by isometries on M.

3 Topics in spectral geometry

A central topic in spectral geometry is the study of the eigenvalue problem

$$-\Delta_g u = \lambda u,$$

where Δ_g is the Laplacian associated to some Riemannian manifold (M, g) and u is a function on M. The Laplacian is an unbounded self-adjoint operator. As we will see later, when the manifold M is compact the spectrum of the Laplacian is discrete. This means that, for any $j \in \mathbb{N}$, there exist a nontrivial function u_j on M and a real number λ_j such that

$$-\Delta_g u_j = \lambda_j u_j.$$

In addition

$$\lambda_j \to \infty \quad \text{as} \quad j \to \infty.$$

The numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$ and the functions $u_1, u_2, u_3, \ldots$ are called the eigenvalues and the eigenfunctions of the Laplacian on (M, g), respectively. The set of all eigenvalues of the Laplacian on (M, g) is called the spectrum of (M, g) and is denoted by $\mathrm{Spec}(M, g)$.

There are relations between the spectrum of (M, g) and the geometry of (M, g). We can investigate these relations in two directions: Given a manifold (M, g), what can we say about the spectrum of (M, g)? Conversely, given the spectrum of (M, g), what can we say about the manifold (M, g)? These questions lead to two types of problems in spectral geometry: *direct problems* and *inverse problems.*

Direct problems. The most natural direct problem is to compute the eigenvalues and eigenfunctions of the Laplacian for a given manifold (M, g). This computation, however, can be performed explicitly only for a few choices of (M, g). We give some examples of exact computations later. In the general case, where explicit exact computations are not available, we must seek for qualitative statements about the spectrum. Two statements of such type are the Rayleigh-Faber-Krahn inequality and the Weyl formula, which we introduce next. These, as well as other classical inequalities and asymptotic results about eigenvalues, will be discussed later.

The Rayleigh-Faber-Krahn inequality. Let Ω be an open subset of $\mathbb{R}^2$ and let u be a function on $\overline{\Omega}$. Consider the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{on } \Omega$$

with the boundary condition

$$u|_{\partial\Omega} = 0,$$

where Δ denotes the Laplacian on Ω and $\partial\Omega$ denotes the boundary of Ω. Let $\lambda_1(\Omega)$ be the first eigenvalue of the above problem. Lord Rayleigh [10] conjectured that, among all open subsets of $\mathbb{R}^2$ with a given area, the disc has the lowest first eigenvalue. In other words

$$\lambda_1(\Omega) \geq \lambda_1(D),$$

where D is an open disc in $\mathbb{R}^2$ with area equal to the area of Ω. Rayleigh made this conjecture based on computations of $\lambda_1(\Omega)$ for specific Ω. This conjecture was proved by Faber [2] and Krahn [7], independently. As we will see later, the Rayleigh-Faber-Krahn inequality generalizes to $\mathbb{R}^n$ for $n \geq 2$.

The Weyl formula. Consider the eigenvalue problem with boundary condition in the previous paragraph, and let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be its eigenvalues. Let λ be a real number. According to Kac [6], the Dutch physicist Lorentz conjectured that the number of eigenvalues less than λ is simply proportional to the area of Ω and λ. In other words, Lorentz conjectured that

$$N(\lambda) \sim \frac{\text{Area}(\Omega)}{2\pi}\lambda,$$

where $N(\lambda)$ denotes the cardinality of the set $\{j \in \mathbb{N} \mid \lambda_j < \lambda\}$ and $\text{Area}(\Omega)$ denotes the area of Ω. The symbol $\sim$ means that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{2\pi}.$$

Lorentz made this conjecture based on computations of the eigenvalues for specific Ω. Weyl proved this conjecture by showing that

$$N(\lambda) \sim \frac{\text{Area}(\Omega)}{2\pi}\lambda \quad \text{as} \quad \lambda \to \infty.$$

This result generalizes to any compact Riemannian manifold (more on this later).

Inverse problems. On a Riemannian manifold (M, g), the Laplacian Δ_g is completely determined by the metric g. Hence, the spectrum of (M, g) is completely determined by g. Thus, if two manifolds (M_1, g_1) and (M_2, g_2) are isometric, they have the same spectrum. In view of this, a natural question arises: Does the spectrum of (M, g) determines (M, g)? A question of this type was investigated by Kac in the famous article "Can one hear the shape of a drum?" [6]. The precise formulation of the question is the following:

Isospectral problem. If two Riemannian manifolds (M_1, g_1) and (M_2, g_2) have the same spectrum, are they isometric? The answer is negative: In 1964, Milnor [9] constructed two manifolds of dimension 16 that have the same spectrum but are not isometric. In the 1980s, Gordon and Wilson [5] and Sunada [11] provided a systematic way of constructing examples of this type. In "Can one hear the shape of a drum?" (1966), Kac considered the isospectral problem for planar domains. In 1992, Gordon, Webb and Wolpert [3] provided the first example of planar domains that have the same spectrum but are not isometric. In the 2010s, many examples of this type are available. Therefore, "One cannot hear the shape of a Drum" [4].

Classical heat invariants. Although the spectrum of a manifold does not determine the manifold, it determines some properties of the manifold. For instance, in "Can one hear the shape of a drum", Kac proved the following formula under the hypothesis that Ω is a convex polygon:

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \sim \frac{1}{4\pi t}\Big(\text{Area}(\Omega) - \sqrt{4\pi t}\,\text{Length}(\partial\Omega) + \frac{2\pi t}{3}(1 - \gamma(\Omega))\Big),$$

where $\text{Length}(\partial\Omega)$ denotes the length of $\partial\Omega$ and $\gamma(\Omega)$ denotes the genus (the number of holes) of Ω. Here, as above, the numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the eigenvalues of the problem $-\Delta u = \lambda u$ on Ω with the boundary condition $u|_{\partial\Omega} = 0$. Using this formula, given the spectrum $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ of Ω, we can deduce the area of Ω, the perimeter of Ω, and the genus of Ω. More generally, as we will see later, the spectrum of a compact manifold (M, g) determines the dimension of (M, g), the volume of (M, g), and the integral of the scalar curvature on (M, g). The above formula also generalizes to any compact Riemannian manifold.

Another question in the context of inverse problems is "What sequences of real numbers can form the spectrum of a manifold?" A simpler version of this question is the following:

Prescribing the spectrum. Given a topological manifold M and a finite increasing sequence of real numbers $a_1, a_2, \ldots, a_n$ for some $n \in \mathbb{N}$, does there exists a metric g such that the first nth elements of the spectrum of (M, g) are equal to $a_1, a_2, \ldots, a_n$, respectively? The answer is positive for compact manifolds of dimension greater than or equal to 3. This was proved by Colin de Verdière in 1987 [1].

References

[1] Y. Colin de Verdière, Construction de laplaciens dont une partie finie du spectre est donnée, *Ann. Sci. Ec. Norm. Sup. Paris* **20**, 599-615 (1987)

[2] G. Faber, Beweis, das unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreissförmige den tiefsten Grundton gibt, *Sitz. ber. bayer. Akad. Wiss.*, 169-172 (1923)

[3] C. Gordon, D. Webb, and S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, *Invent. mathematicae* **110**, 1-22 (1992)

[4] C. Gordon, D. Webb, and S. Wolpert, One cannot hear the shape of a Drum, *Bulletin AMS* **27**, 134-138 (1992)

[5] C. Gordon and E. N. Wilson, Isospectral deformations on compact manifolds, *J. Diff. Geom.* **19**, 241-256 (1984)

[6] M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* **73**(4), 1-23 (1966)

[7] E. Krahn, Ueber eine von Rayleight formulierte Minimaleigenschaft des Kreises, *Math. Ann.* **94**, 97-100 (1924)

[8] O. Lablée, *Spectral Theory in Riemannian Geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2015.

[9] J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, *Proceedings of the National Academy of Sciences of the United States of America* **51**, 542 (1964)

[10] L. Rayleight, *The theory of sound*, 2nd edition, London, 1894/96.

[11] T. Sunada, Riemannian covering and isospectral manifolds, *Ann. of Math.* **121**, 169-186 (1985)