Lecture 2
Operators on Hilbert spaces

1 Hilbert spaces

We denote by $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space over the field $\mathbb{C}$ (or $\mathbb{R}$). We adopt the convention that the inner product on H is linear in the first variable and conjugate linear in the second variable. The induced norm on H is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x \in H$. The normed vector space $(H, \|\cdot\|)$ is a Banach space. That is, the metric space (H, d) is complete, where d denotes the induced metric $d(x, y) = \|x - y\|$ for $x, y \in H$. When H is a Hilbert space over the field $\mathbb{R}$, the conjugate linearity and conjugate symmetry of the inner product reduce to linearity and symmetry, respectively.

We often suppose that H is a separable Hilbert space. This means that H contains a countable dense set. A separable Hilbert space H contains a countable orthonormal basis $\{e_1, e_2, e_3, \dots\}$. Thus, for every $x \in H$ we have $x = \sum_{j=1}^{\infty} c_j e_j$, where $c_1, c_2, c_3, \dots$ are constants.

A basic inequality on Hilbert spaces is the Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \quad \text{for all } x, y \in H.$$

Example 1. (Space l^2) We define l^2 to be the set of all sequences of complex numbers $(x_1, x_2, x_3, \dots)$ such that $\sum_{j=1}^{\infty} |x_j|^2 < \infty$. It is common to use the notation $x = (x_j)_{j\in\mathbb{N}}$ to represent a sequence in l^2. The set l^2 with the inner product defined by $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$ for $x, y \in l^2$ is a Hilbert space.

Example 2. (Space $L^2([a, b])$) Let $[a, b]$ be an interval of $\mathbb{R}$. We define $L^2([a, b])$ to be the set of all functions $f : [a, b] \to \mathbb{C}$ such that $\int_a^b |f(x)|^2 dx < \infty$. The set $L^2([a, b])$ with the inner product defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

for $f, g \in L^2([a, b])$ is a Hilbert space.

Example 3. (Space $H^1((a, b))$) For $(a, b) \subset \mathbb{R}$, we define $H^1((a, b))$ to be the set of all functions $f \in L^2([a, b])$ such that $f' \in L^2([a, b])$. The statement

$f' \in L^2([a,b])$ means that there exists $g \in L^2([a,b])$, denoted by f', such that

$$\int_a^b f(x)u'(x)\,dx = -\int_a^b g(x)u(x) \quad \text{for all } u \in C_c^\infty((a,b)),$$

where $C_c^\infty((a,b))$ denotes the set of all functions in $C^\infty((a,b))$ whose support is compact and contained in (a,b). The set $H^1((a,b))$ with the inner product defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx + \int_a^b f'(x)\overline{g'(x)}\,dx$$

for $f, g \in H^1((a,b))$ is a Hilbert space.

2 Bounded operators

A bounded operator on H is a linear mapping $T : H \to H$ such that

$$\|Tx\| \le C\|x\| \quad \text{for all } x \in H$$

for some constant $C > 0$. The least of such numbers C, denoted by $\|T\|$, is called the operator norm of T:

$$\|T\| = \sup_{\substack{x\in H\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in H\\ \|x\|=1}} \|Tx\| = \sup_{\substack{x,y\in H\\ \|x\|=\|y\|=1}} |\langle Tx, y\rangle|.$$

If S and T are bounded operators on H, we have

$$\|ST\| \le \|S\|\|T\|.$$

We denote by $B(H)$ the set of all bounded operators on H. The set $B(H)$ is a vector space and an algebra. The pair $(B(H), \|\cdot\|)$ is a Banach space.

Theorem 1. *A linear operator $T : H \to H$ is bounded if and only if T is continuous.*

Example 4. Let $k \in L^2([a,b]\times[a,b])$. The linear mapping

$$T : \begin{cases} L^2([a,b]) \to L^2([a,b]) \\ f(\cdot) \mapsto \int_a^b k(\cdot, y) f(y)\,dy \end{cases}$$

is a bounded operator. Furthermore $\|T\| \le \|k\|_{L^2([a,b]\times[a,b])}$.

A bounded operator $T : H \to H$ is said to be invertible if there exists $S \in B(H)$, denoted by T^{-1}, such that $TS = I$ and $ST = I$. (We observe that, by the open mapping theorem, if $T : H \to H$ is bounded and bijective, then T^{-1} is bounded.)

Theorem 2. *Let $T \in B(H)$. For all $z \in \mathbb{C}$ such that $|z| > \|T\|$, the operator $zI - T$ is invertible and*

$$(zI - T)^{-1} = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} T^j.$$

Finally, we mention that if the Hilbert space H has finite dimension, then every linear operator on H is bounded.

3 Bounded linear functionals

A bounded linear functional on H is a linear mapping $l : H \to \mathbb{C}$ such that

$$|l(x)| \le C\|x\| \quad \text{for all } x \in H$$

for some constant $C > 0$. The least of such numbers C, denoted by $\|l\|$, is called the operator norm of l:

$$\|l\| = \sup_{\substack{x \in H \\ x \neq 0}} \frac{|l(x)|}{\|x\|} = \sup_{\substack{x \in H \\ \|x\|=1}} |l(x)|.$$

We denote by H^* the set of all bounded linear functionals on H. The set H^* is a vector space. The pair $(H^*, \|\cdot\|)$ is a Banach space.

Theorem 3. *A linear functional $l : H \to \mathbb{C}$ is bounded if and only if l is continuous.*

Every element $y \in H$ determines a bounded linear functional defined by $f(x) = \langle x, y \rangle$ for $x \in H$. (Note that $|f(x)| \le \|y\|\|x\|$ and consequently $f \in H^*$.) Conversely, every bounded linear functional on H has the form $x \mapsto \langle x, y \rangle$ for some $y \in H$:

Theorem 4 (Riesz-Frechet representation theorem)**.** *If $l \in H^*$, there exists a unique $y \in H$ such that*

$$l(x) = \langle x, y \rangle \quad \textit{for all } x \in H.$$

In addition $\|l\| = \|y\|$.

The Riesz-Frechet theorem has the following useful generalization:

Theorem 5 (Lax-Milgram)**.** *Let H be a Hilbert space and suppose that $B : H \times H \to \mathbb{C}$ satisfies the following properties:*

1. *B is linear in the first variable and conjugate linear in the second variable.*

2. There exists $C > 0$ such that $|B(x,y)| \le C\|x\|\|y\|$ for all $x, y \in H$.

3. There exists $M > 0$ such that $|B(x,x)| \ge M\|x\|^2$ for all $x \in H$.

Then, if $l \in H^*$, there exists a unique $y \in H$ such that

$$l(x) = B(x,y) \quad \text{for all } x \in H.$$

In addition $\|y\| \le M^{-1}\|l\|$.

Remark 1.

- The Lax-Milgram Theorem holds true in the real setting. That is, the theorem holds true if we replace H by a Hilbert space over $\mathbb{R}$ and B by a bilinear real-valued function.
- The function B in the Lax-Milgram Theorem does not have to be conjugate symmetric (or symmetric in the real setting).

4 Unbounded operators

We have seen the notion of bounded operator $T : H \to H$. More generally, an operator on H is a linear mapping

$$T : \begin{cases} D(T) \to H \\ x \mapsto Tx, \end{cases}$$

where $D(T)$, called the domain of T, is a vector subspace of H (usually dense in H). (Recall that X is dense in H if $\overline{X} = H$.)

Let $S : D(S) \to H$ and $T : D(T) \to H$ be operators on H. We say that T is an extension of S, denoted $S \subset T$, if

$$D(S) \subset D(T) \quad \text{and} \quad T|_{D(S)} = S.$$

Consider an operator $T : D(T) \to H$. The kernel of T, denoted by $\mathrm{Ker}(T)$, is defined by

$$\mathrm{Ker}(T) = \{x \in D(T) \mid Tx = 0\}.$$

The range of T, denoted by $\mathrm{Ran}(T)$, is defined by

$$\mathrm{Ran}(T) = \{Tx \in H \mid x \in D(T)\}.$$

Example 5. Let $T : D(T) \to L^2(\mathbb{R})$ be the operator defined by

$$(Tf)(x) = xf(x) \quad \text{for } x \in \mathbb{R},$$

where $D(T) = \{f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |xf(x)|^2\,dx < \infty\}$. The domain $D(T)$ is a subspace of $L^2(\mathbb{R})$ with $D(T) \ne L^2(\mathbb{R})$.

5 Closed operators

The graph of an operator $T : D(T) \to H$, denoted by $\Gamma(T)$, is defined by

$$\Gamma(T) = \{(x, Tx) \in H \times H \mid x \in D(T)\}.$$

An operator $T : D(T) \to H$ is said to be closed if $\Gamma(T)$ is a closed subset of the product space $H \times H$. That is, T is closed if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} Tx_n = y \quad \text{imply} \quad (x, y) \in \Gamma(T).$$

Theorem 6. *An operator $T : D(T) \to H$ is closed if and only if $D(T)$ is a Banach space with respect to the norm $\|x\|_T = (\|x\| + \|Tx\|)^{1/2}$.*

We observe that, if $D(T) = H$, then

$$\begin{aligned} T \text{ bounded} &\implies T \text{ closed} \quad \text{(by a simple argument)},\\ T \text{ closed} &\implies T \text{ bounded} \quad \text{(by the closed graph theorem)}. \end{aligned}$$

Thus we have the following theorem:

Theorem 7. *Let $T : D(T) \to H$ be an operator on H. If $D(T) = H$, then T is closed if and only if T is bounded.*

Example 6. For $k = 1, \dots, 5$, consider the operators

$$T_k : \begin{cases} D(T_k) \to L^2([a, b]) \\ f \mapsto if' \end{cases}$$

with

$$\begin{aligned} &D(T_1) = H^1((a, b)),\\ &D(T_2) = \{f \in H^1((a, b)) \mid f(a) = 0\},\\ &D(T_3) = \{f \in H^1((a, b)) \mid f(b) = 0\},\\ &D(T_4) = \{f \in H^1((a, b)) \mid f(a) = f(b)\},\\ &D(T_5) = H_0^1((a, b)) = \{f \in H^1((a, b)) \mid f(a) = f(b) = 0\}. \end{aligned}$$

The operators $T_1, \dots, T_5$ are closed with dense domains. (We observe that the Sobolev embedding theorem implies that $H^1((a, b)) \subset C([a, b])$. Thus it makes sense to use the pointwise values of the functions in the definitions of $D(T_2), \dots, D(T_5)$.)

6 Adjoint operators

Let $T : D(T) \to H$ be an operator with $D(T)$ dense on H. The adjoint of T, denoted by T^*, is the operator defined by

$$\begin{aligned} y \in D(T^*) \quad &\iff \quad \exists \eta \in H \text{ such that } \langle Tx, y\rangle = \langle x, \eta\rangle \text{ for all } x \in D(T),\\ T^* y = \eta \quad &\iff \quad \langle Tx, y\rangle = \langle x, \eta\rangle \text{ for all } x \in D(T). \end{aligned}$$

By the properties that we present below, the above formulae indeed define an operator. Thus we can write

$$T^* : \begin{cases} D(T^*) \to H \\ x \mapsto T^*x := \eta. \end{cases}$$

We mention the following properties:

(i) η is unique.

(ii) By the Riesz-Frechet Theorem, we have $y \in D(T^*)$ if and only if there exists $C_y > 0$ such that $|\langle Tx, y\rangle| \leq C_y \|x\|$ for all $x \in D(T)$.

(iii) $0 \in D(T^*)$.

(iv) If $S \subset T$, then $T^* \subset S^*$.

(v) T^* is closed (even if T is not closed).

(vi) If T is closed, then $T^{**} = T$.

Let us prove the uniqueness of η: Suppose that there exists $\eta_1, \eta_2 \in D(T)$ such that $\langle Tx, y\rangle = \langle x, \eta_1\rangle$ for all $x \in D(T)$ and $\langle Tx, y\rangle = \langle x, \eta_2\rangle$ for all $x \in D(T)$. Then we obtain $\langle x, \eta_1 - \eta_2\rangle = 0$ for all $x \in D(T)$. This implies that $\eta_1 - \eta_2 \in D(T)^\perp$. On the other hand, since $D(T)$ is dense in H, we have $D(T)^\perp = \{0\}$ because $\overline{D(T)} = H$ and $H = \overline{D(T)} \oplus D(T)^\perp$ since $\overline{D(T)}^\perp = D(T)^\perp$. Therefore $\eta_1 - \eta_2 = 0$, which proves the claim.

Example 7. For $k = 1, \ldots, 5$, consider the operators

$$T_k : \begin{cases} D(T_k) \to L^2([a,b]) \\ f \mapsto if' \end{cases}$$

with

$$\begin{aligned} D(T_1) &= H^1((a,b)), \\ D(T_2) &= \{f \in H^1((a,b)) \mid f(a) = 0\}, \\ D(T_3) &= \{f \in H^1((a,b)) \mid f(b) = 0\}, \\ D(T_4) &= \{f \in H^1((a,b)) \mid f(a) = f(b)\}, \\ D(T_5) &= H^1_0((a,b)) = \{f \in H^1((a,b)) \mid f(a) = f(b) = 0\}. \end{aligned}$$

We have seen in Example 6 that these operators are closed. In addition we have $T_1^* = T_5$, $T_2^* = T_3$, $T_3^* = T_2$, $T_4^* = T_4$, and $T_5^* = T_1$.

Theorem 8. *If $T : D(T) \to H$ is an operator with $D(T)$ dense in H, then*

$$\mathrm{Ker}(T^*) = (\mathrm{Ran}(T))^\perp.$$

Theorem 9. *Let $T : H \to H$ be a bounded operator. Then $D(T^*) = H$ and the operator T^* is bounded with $\|T^*\| = \|T\|$.*

7 Symmetry and self-adjointness

Consider an operator $T : D(T) \to H$ with $D(T)$ dense in H.

The operator T is said to be symmetric if

$$\langle x, Ty \rangle = \langle Tx, y \rangle \quad \text{for all } x, y \in D(T).$$

Equivalently, T is symmetric if $T \subset T^*$. In particular, if T is symmetric, then $\langle x, Tx \rangle \in \mathbb{R}$ for all $x \in D(T)$.

The operator T is said to be self-adjoint if

$$T^* = T.$$

We observe that

$$\begin{aligned} T \text{ self-adjoint } &\implies T \text{ symmetric}, \\ T \text{ symmetric and } D(T^*) \subset D(T) &\implies T \text{ self-adjoint}. \end{aligned}$$

If T is a bounded operator on H, then T is symmetric if and only if T is self-adjoint.

Theorem 10. *Suppose that T is a bounded self-adjoint operator on H. Then*

$$\|T\| = \sup_{\substack{x \in H \\ \|x\|=1}} |\langle Tx, x \rangle|.$$

Theorem 11. *Let $T : D(T) \to H$ be an operator with $D(T)$ dense in H. If T is self-adjoint, then T is closed.*

Theorem 12. *Let $T : D(T) \to H$ be a symmetric operator. The following properties are equivalent:*

(i) *T is self-adjoint.*

(ii) *T is closed and* $\operatorname{Ker}(T^* \pm iI) = \{0\}$.

(iii) $\operatorname{Ran}(T \pm iI) = H$.

References

[1] O. Lablée, *Spectral Theory in Riemannian Geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2015.