Lecture 3
Spectrum of operators

1 Resolvent set, spectrum, point spectrum

As usual, we denote by H a Hilbert space. Let $T : D(T) \to H$ be a closed linear operator with $D(T)$ dense in H.

The resolvent set of T, denoted $\rho(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is a bijection (from $D(T)$ to H) and $(\lambda I - T)^{-1} : H \to H$ is a bounded operator.

The spectrum of T, denoted $\sigma(T)$, is the set $\mathbb{C} \setminus \rho(T)$.

By definition, we have $\rho(T) \cup \sigma(T) = \mathbb{C}$ and $\rho(T) \cap \sigma(T) = \emptyset$. In addition, $\rho(T)$ is an open subset of $\mathbb{C}$, and consequently $\sigma(T)$ is a closed subset of $\mathbb{C}$.

In order to study the spectrum of T, it is useful to classify subsets of $\sigma(T)$ according to the reason why its elements fail to be in the resolvent set. There are different ways of doing such classification (and we will not talk much about this). In this course, we are particularly interested in the subset of $\sigma(T)$ (called point spectrum of T) that consists of eigenvalues of T (in the classical sense of linear algebra). Here is the definition of point spectrum:

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if $\lambda I - T$ is not injective, that is, $\mathrm{Ker}(\lambda I - T) \neq \{0\}$. The point spectrum of T, denoted $\sigma_p(T)$, is the set of all eigenvalues of T.

Clearly $\sigma_p(T) \subset \sigma(T)$.

Let $\lambda \in \sigma(T)$. A vector $x \in D(T)$ such that $Tx = \lambda x$ and $x \neq 0$ is called an eigenvector of T associated to λ.

Example 1. Suppose that M is an $n \times n$ matrix that is diagonalizable (this is the case if M is symmetric or hermitian). Then there exists a matrix V such that $V^{-1}MV = D$, where D is a diagonal matrix. The diagonal entries of D are the eigenvalues of M. The columns of V are the corresponding eigenvectors of M. In this example $\sigma(T) = \sigma_p(T)$.

More generally, if $T : H \to H$ is a bounded operator and $\dim(H) < \infty$, then $\sigma(T) = \sigma_p(T)$.

Example 2 (Multiplication operator)**.** Let (X, Σ, μ) be a σ-finite measure space, and let $F : X \to \mathbb{C}$ be a measurable function that is finite almost everywhere on X. Set

$$M_F : \begin{cases} D(M_F) \to L^2(X,\mu) \\ u \mapsto Fu \end{cases}$$

with $D(M_F) = \{u \in L^2(X,\mu) \mid Fu \in L^2(X,\mu)\}$. We have the following properties:

(i) $D(M_F)$ is dense in $L^2(X,\mu)$.

(ii) If F is real-valued, then M_F is self-adjoint.

(iii) If F is bounded, then M_F is bounded on $L^2(X,\mu)$ and

$$\|M_F\| = \|F\|_{L^\infty(X,\mu)}.$$

Furthermore

$$\rho(M_F) = \{\lambda \in \mathbb{C} \mid \exists\, \varepsilon > 0 \text{ such that } |\lambda - F(x)| \geq \varepsilon \text{ almost everywhere}\}$$

and

$$\sigma_p(M_F) = \{\lambda \in \mathbb{C} \mid \mu(\{x \in X \mid F(x) = \lambda\}) > 0\}.$$

For instance, if $(X,\Sigma,\mu) = (\mathbb{R},\mathcal{B},dx)$ and $F(x) = x$, we obtain $\sigma_p(M_F) = \emptyset$.

2 Resolvent operator

Let $T : D(T) \to H$ be an operator. The resolvent of T at $\lambda \in \rho(T)$, denoted $R_\lambda(T)$, is the operator defined by

$$R_\lambda(T) = (\lambda I - T)^{-1}.$$

It follows from the definition that $R_\lambda(T)$ is a bounded operator on H.

The resolvent of T satisfies the following identities for all $\lambda, \mu \in \rho(T)$:

- $R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\lambda(T) R_\mu(T)$.
- $R_\lambda(T) R_\mu(T) = R_\mu(T) R_\lambda(T)$.

In addition, the mapping

$$\begin{cases} \rho(T) \to B(H) \\ \lambda \mapsto R_\lambda(T) \end{cases}$$

is analytic. (Recall that $B(H)$ denotes the set of all bounded operators on H.) The notion of analytic function from complex analysis generalizes to functions with values in Banach spaces. Many features of the theory of analytic functions generalizes as well. Here, we just mention one definition that characterizes analytic functions in a general setting: Let Ω be an open subset of $\mathbb{C}$ and let X be a Banach space. A mapping $f : \Omega \to X$ is said to be analytic at $z_0 \in \Omega$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If $f : \Omega \to X$ is analytic at all $z_0 \in \Omega$, then f is said to be analytic.

Theorem 1. *Let T be a bounded operator on H. For all $z \in \mathbb{C}$ such that $|z| > \|T\|$, the operator $(zI - T)$ is bijective, the operator $(zI - T)^{-1}$ is bounded, and*

$$(zI - T)^{-1} = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} T^k.$$

Since the mapping $\lambda \mapsto R_\lambda(T)$ is analytic, we can use the theory of analytic functions to study this mapping and obtain information about the spectrum of T. The proof of the next proposition illustrates this idea. It uses the above theorem and properties of analytic functions.

Proposition 1. *Let T be a bounded operator on H. Then $\sigma(T) \neq \emptyset$.*

Idea of proof. By contradiction. Suppose that $\sigma(T) = \emptyset$. Then $R_\lambda(T)$ is an entire analytic function. In addition, for $|z| > \|T\|$, we have

$$R_\lambda(T) = \frac{1}{z} \sum_{j=0} \left(\frac{T}{z} \right)^j.$$

Thus $\|R_\lambda(T)\| \to 0$ as $|z| \to \infty$. Hence, by Liouville's Theorem, we would have $R_\lambda(T) = 0$ for all $z \in \mathbb{C}$. This is a contradiction because $R_\lambda(T) \neq 0$. Therefore $\sigma(T) \neq \emptyset$. □

Theorem 2. *Let $T : D(T) \to H$ be a closed operator such that $\rho(T) \neq \emptyset$. For any $\lambda_0 \in \rho(T)$, we have*

$$\sigma_p(R_{\lambda_0}) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - \mu} \;\middle|\; \mu \in \sigma_p(T) \right\}$$

and

$$\sigma(R_{\lambda_0}) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - \mu} \;\middle|\; \mu \in \sigma(T) \right\}.$$

Proof of the first equality. Suppose that $\lambda \in \sigma_p(R_{\lambda_0}(T))$ with $\lambda \neq 0$. Then there exists $\varphi \in H$ such that $(\lambda_0 I - T)^{-1} \varphi = \lambda \varphi$ and $\varphi \neq 0$. Since $(\lambda_0 I - T)$ is a bijection from $D(T)$ to H, we conclude that $\varphi \in D(T)$. Calculating, we obtain

$$\begin{aligned} T\varphi &= T\lambda^{-1}(\lambda_0 I - T)^{-1}\varphi \\ &= -\lambda^{-1}(\lambda_0 I - T - \lambda_0 I)(\lambda_0 I - T)^{-1}\varphi \\ &= \lambda^{-1}(\lambda_0 R_{\lambda_0}(T) - I)\varphi \\ &= \lambda^{-1}(\lambda_0 \lambda - I)\varphi \\ &= \mu\varphi \end{aligned}$$

with $\mu = \lambda_0 - \lambda^{-1}$. Thus $\mu \in \sigma_p(T)$ and $\lambda = (\lambda_0 - \mu)^{-1}$. Therefore

$$\sigma_p(R_{\lambda_0}(T)) \setminus \{0\} \subset \{(\lambda_0 - \mu)^{-1} \mid \mu \in \sigma_p(T)\}.$$

Similarly we prove that

$$\{(\lambda_0 - \mu)^{-1} \mid \mu \in \sigma_p(T)\} \subset \sigma_p(R_{\lambda_0}(T)) \setminus \{0\}\,.$$

This proves the first equality of the theorem. □

3 The self-adjoint case

Theorem 3. *Let $T : D(T) \to H$ be a self-adjoint operator. We have the following properties:*

(i) $\sigma(T) \subset \mathbb{R}$.

(ii) *Eigenvectors associated to different eigenvalues are orthogonal.*

To prove the Part (i) of the above theorem, we will use the following proposition:

Proposition 2. *Let $T : D(T) \to H$ be a self-adjoint operator. If there exists $C > 0$ such that $\|(\lambda I - T)u\| \geq C\|u\|$ for all $u \in D(T)$, then $\lambda \in \rho(T)$.*

Proof of Theorem 3. (i) For $z = \lambda + i\mu$ with $\mu \neq 0$, we have

$$\begin{aligned} \|(zI - T)u\|^2 &= \|(\lambda I - T)u\|^2 + |\mu|^2\|u\|^2 \\ &\geq |\mu|^2\|u\|^2 \end{aligned}$$

for all $u \in D(T)$. By the above proposition, we conclude that $z \in \rho(T)$. Since $\rho(T)$ and $\sigma(T)$ are disjoint, we have $z \in \sigma(T)$ only if $\Re(z) = 0$. Therefore $\sigma(T) \subset \mathbb{R}$.

(ii) Let u and v be eigenvectors of T associated to different eigenvalues λ and η, respectively. Then

$$\lambda\langle u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \langle u, v\rangle\eta.$$

Hence $\langle u, v\rangle = (\lambda - \eta)^{-1}0 = 0$. Thus u and v are orthogonal. □

Theorem 4. *Let T be a bounded self-adjoint operator on H. Define*

$$m = \inf_{\substack{u \in H \\ \|u\|=1}} \langle Tu, u \rangle \qquad \text{and} \qquad M = \sup_{\substack{u \in H \\ \|u\|=1}} \langle Tu, u \rangle.$$

We have $\sigma(T) \subset [m, M]$ and $m, M \in \sigma(T)$.

Proposition 3. *Let T be a bounded self-adjoint operator on H. We have*

$$\|T\| = \sup_{\substack{u \in H \\ \|u\|=1}} |\langle Tu, u \rangle|.$$

Corollary 1. *Let T be a bounded self-adjoint operator on H. If $\sigma(T) = \{0\}$, then $T = 0$.*

Proof. Apply the theorem and the proposition above. □

4 The spectral theorem

Theorem 5 (Spectral theorem—Multiplication operator form)**.** *Suppose that $T : D(T) \to H$ is a self-adjoint operator. There exist*

1. *a measure space (X, μ),*
2. *a measurable function $F : X \to \mathbb{R}$,*
3. *a unitary operator $U : H \to L^2(X, \mu)$*

such that

$$UTU^{-1} = M_F,$$

where M_F is the multiplication operator

$$M_F : \begin{cases} D(M_F) \to L^2(X, \mu) \\ u \mapsto Fu \end{cases}$$

with $D(M_F) = \{u \in L^2(X, \mu) \mid Fu \in L^2(X, \mu)\}$. If H is separable, then μ can be chosen to be a finite measure.

References

[1] O. Lablée, *Spectral Theory in Riemannian Geometry*, EMS Textbooks in Mathematics, European Mathematical Society, 2015.

[2] P. D. Hislop and I. M. Sigal, *Introduction to Spectral Theory With Application to Schrödinger Operator*, Applied Mathematical Sciences 113, Springer, 1996.