Problem Set 1 – Solutions

Problem 1. Let $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the linear differential operator of order m defined by

$$L = \sum_{|\alpha|_1 \leq m} a_\alpha \partial^\alpha,$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$ for every multi-index α.

(a) ($\Rightarrow$) Suppose that the function a_α is constant for every α. For $u \in \mathbb{R}^n$, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the translation defined by $T(x) = x + u$. We observe that (for any multi-index β)

$$\begin{aligned}
\frac{\partial}{\partial x_j}(\partial^\beta f \circ T) &= \sum_{k=1}^n \Big(\frac{\partial}{\partial x_k}\partial^\beta f\Big) \circ T \,\frac{\partial}{\partial x_j} T_k \\
&= \Big(\frac{\partial}{\partial x_j}\partial^\beta f\Big) \circ T = (\partial^{(\beta_1,\ldots,\beta_j+1,\ldots,\beta_n)} f) \circ T.
\end{aligned}$$

Consequently

$$\begin{aligned}
\partial^\alpha(f \circ T) &= \partial^{(\alpha_1,\ldots,\alpha_j-1,\ldots,\alpha_n)} \frac{\partial}{\partial x_j}(f \circ T) \\
&= \partial^{(\alpha_1,\ldots,\alpha_j-1,\ldots,\alpha_n)} \Big(\frac{\partial}{\partial x_j} f\Big) \circ T \\
&\;\;\vdots \\
&= (\partial^\alpha f) \circ T.
\end{aligned}$$

Hence

$$\begin{aligned}
(Lf) \circ T &= \Big(\sum a_\alpha \partial^\alpha f\Big) \circ T \\
&= \sum (a_\alpha \partial^\alpha f) \circ T \\
&= \sum (a_\alpha \circ T)(\partial^\alpha f) \circ T \\
&= \sum a_\alpha \partial^\alpha (f \circ T) \\
&= L(f \circ T).
\end{aligned}$$

Thus L is invariant by translations.

($\Leftarrow$) Conversely, suppose that L is invariant by translations:

$$(Lf) \circ T = L(f \circ T).$$

This implies that

$$0 = \sum_{\alpha} (a_\alpha \circ T - a_\alpha)(\partial^\alpha f) \circ T$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Let $\phi(x) = x^\beta$. A simple calculation shows that

$$(\partial \phi)(x) = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta \end{cases}.$$

Hence, if $\alpha \leq \beta$,

$$(\partial^\alpha \phi) \circ T = \frac{\beta!}{(\beta-\alpha)!}(x+u)^{\beta-\alpha} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}.$$

Let J be a function in $C_c^\infty(\mathbb{R}^n)$ such that $0 \leq J(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $J(x) = 1$ for all x in the ball of radius $2|u|$ with center at 0. Using the above observation with $f = J\phi$, we obtain

$$0 = \sum_{\alpha} (a_\alpha \circ T - a_\alpha)(\partial^\alpha f) \circ T(-u) = (a_\beta \circ T - a_\beta)(-u).$$

This implies that $a_\beta(0) - a_\beta(-u) = 0$ for all $u \in \mathbb{R}^n$ and β with $|\beta| \leq m$. That is, the coefficients of the operator L are constant.

(b) We observe that

$$L = \sum_{\alpha} a_\alpha \partial^\alpha = \sum_{\alpha} (2\pi i)^{|\alpha|_1} \frac{1}{(2\pi i)^{|\alpha|_1}} \partial^\alpha = \sum_{\alpha} b_\alpha D^\alpha$$

where $b_\alpha = (2\pi i)^{|\alpha|_1} b_\alpha$ and $D^\alpha = (2\pi i)^{-|\alpha|_1} \partial^\alpha$. Thus we may write

$$L = P(D)$$

where P is the polynomial $P(\xi) = \sum_\alpha b_\alpha \xi^\alpha$.

(c) Let T be a rotation and suppose that L has constant coefficients. On the one hand, using Properties (ii) and (iii) of Theorem 2, we find that

$$\mathcal{F}\big((Lf) \circ T\big)(\xi) = (P \circ T)(\xi)(\mathcal{F}f) \circ T(\xi).$$

On the other hand, using Property (ii), we obtain

$$\mathcal{F}\big(L(f \circ T)\big)(\xi) = P(\xi)(\mathcal{F}f) \circ T(\xi).$$

Thus, the operator L is invariant by rotations, that is,

$$(Lf) \circ T = L(F \circ T)$$

if and only if

$$P \circ T = P,$$

that is, the polynomial P is invariant by rotations.

(d) Let $P = \sum_{j=0}^{m} P_j$, where P_j is a homogeneous polynomial of degree j, and let T be a rotation.

($\Leftarrow$) If $P_j \circ T = P_j$ for every j, then $P \circ T = P$.

($\Rightarrow$) Conversely, suppose that $P \circ T = P$. We will prove that $P_j \circ T = P_j$ for every j. For $j = 0$, we have

$$\begin{aligned} P_0(\xi) &= \lim_{r\to 0} \sum_{k=0}^{m} P_k(r\xi) = \lim_{r\to 0} P(r\xi) = \lim_{r\to 0} P(rT\xi) = \lim_{r\to 0} \sum_{k=0}^{m} P_k(rT\xi) \\ &= P_0(T\xi). \end{aligned}$$

Thus $P_0 = P_0 \circ T$. Suppose that $P_j \circ T = P_j$ for all $j \in \{0, 1, \ldots, l\}$. Then

$$\begin{aligned} P_{l+1}(\xi) &= \lim_{r\to 0} \frac{1}{r^{l+1}} \sum_{k=l+1}^{m} P_k(r\xi) = \lim_{r\to 0} \frac{1}{r^{l+1}} \Big(P(r\xi) - \sum_{k=0}^{l} P_k(r\xi) \Big) \\ &= \lim_{r\to 0} \frac{1}{r^{l+1}} \Big(P(rT\xi) - \sum_{k=0}^{l} P_k(rT\xi) \Big) \\ &= \lim_{t\to 0} \frac{1}{r^{l+1}} \sum_{k=l+1}^{m} P_k(rT\xi) \\ &= P_{l+1}(T\xi). \end{aligned}$$

Hence $P_{l+1} = P_{l+1} \circ T$. Therefore, by induction, $P_j = P_j \circ T$ for every j.

(e) Let e be a vector in $\mathbb{R}^n$ with $|e| = 1$. For all $x \in \mathbb{R}^n$, there exists a rotation T such that $x = |x|Te$. Since P_j is homogeneous of degree j and rotation-invariant, we have

$$P_j(x) = P_j(|x|Te) = |x|^j P_j(Te) = |x|^j P_j(e).$$

Since $P_j(e)$ is a constant and $|x|^j$ is a polynomial if and only if j is even, it follows that P_j is a polynomial if and only if $P_j(e) = 0$ for every $x \in \mathbb{R}$ if j is odd. Thus

$$P(\xi) = \sum_{j=0}^{m} P_j(\xi) = \sum_{k=0}^{\lfloor m/2 \rfloor} P_{2k}(e)|\xi|^{2k} = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{P_{2k}(e)}{(-4\pi^2)^k} (-4\pi^2|\xi|^2)^k.$$

Therefore $L = P(D) = \sum_{k=0}^{\lfloor m/2 \rfloor} c_k \Delta^k$ with $c_k = P_{2k}(e)/(-4\pi^2)^k$.