Problem Set 2 – Solutions

Problem 1. (a) ($\Leftarrow$) Suppose that $(a_j)_{j\in\mathbb{N}}$ is bounded, that is, $|a_j| \le C$ for all $j \in \mathbb{N}$. Then

$$\begin{aligned}\|M_a x\|_{l^2}^2 &= \|(a_1x_2, a_2x_2, \dots)\|_{l^2}^2 \\ &= \sum_{j\in\mathbb{N}} |a_j|^2|x_j|^2 \le \sup_{j\in\mathbb{N}} |a_j|^2 \sum_{j\in\mathbb{N}} |x_j|^2 \le C^2\|x\|_{l^2}^2.\end{aligned}$$

This implies that $\|M_a\| \le C$. Thus M_a is bounded.

($\Rightarrow$) Conversely, suppose that M_a is bounded, that is, $\|M_a\| \le C$. Then

$$\frac{\|M_a x\|}{\|x\|} \le C$$

for all $x \in l^2$ with $x \ne 0$. For $j \in \mathbb{N}$, let e_j be the sequence whose jth term is equal to 1 and all other terms are equal to 0. Evaluating the above inequality at $x = e_j$, we obtain $|a_j| \le C$. Consequently $(a_j)_{j\in\mathbb{N}}$ is bounded.

(b) As in Part (a), we have $\|M_a\| \le \sup_{j\in\mathbb{N}} |a_j|$ and

$$\|M_a\| \ge \sup_{j\in\mathbb{N}} \frac{\|M_a e_j\|_{l^2}}{\|e_j\|_{l^2}} = \sup_{j\in\mathbb{N}} |a_j|.$$

Therefore $\|M_a x\| = \sup_{j\in\mathbb{N}} |a_j|$.

Problem 2. The operator M is injective but not surjective. Thus M is not bijective. We first show that M is injective. We observe that $Mx = 0$ if and only if $j^{-1}x_j = 0$ for all $j \in N$ if and only if $x_j = 0$ for all $j \in \mathbb{N}$. Hence the kernel of M contains only the zero vector. Thus M is injective. To see that M is not surjective, consider the sequence $(j^{-1})_{j\in\mathbb{N}} \in l^2$. Then $Mx = (j^{-1})_{j\in l^2}$ if and only if $x = (1)_{j\in\mathbb{N}}$. But $(1)_{j\in\mathbb{N}} \notin l^2$. Thus M is not surjective.

Problem 3. (a) By a theorem we saw in class, the operator T_1 is closed if and only if $D(T_1)$ is complete with respect to the norm

$$\|f\|_{T_1} = \|f\|_2 + \|T_1 f\|_2 = \|f\|_2 + \|f'\|_2.$$

The norm $\|f\|_{T_1}$ is equivalent to the norm $\|f\|_{H^1((a,b))} = (\|f\|_2 + \|f'\|_2)^{1/2}$. Thus T_1 is closed if and only if $D(T_1)$ is complete with respect to the norm $\|f\|_{H_1((a,b))}$. But $D(T_1) = H^1((a,b))$, and $H^1((a,b))$ is a Banach space, thus

$((D(T_1), H^1((a,b))$ is complete. Therefore T_1 is closed. The operator T_1 is densely defined because $C_c^\infty((a,b)) \subset D(T_1)$.

(b) As in Part (a), it suffices to prove that $D(T_2)$ is complete with respect to the norm $\|\cdot\|_{H_1((a,b))}$ (which is equivalent to the norm $\|\cdot\|_{T_2}$). We observe that $D(T_2) \subset H^1((a,b))$. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $D(T_2)$. Since $H^1((a,b))$ is closed, the sequence $(f_j)_{j\in\mathbb{N}}$ converges to $f \in H_1((a,b))$. By Theorem 1 (Sobolev embedding), the sequence $(f_j)_{j\in\mathbb{N}}$ converges uniformly to f:

$$\|f_j - f\|_u \leq C\|f_j - f\|_{H^1((a,b))} \longrightarrow 0 \quad \text{as } j \to \infty.$$

Thus

$$|f(0)| = |f_j(0) - f(0)| \leq \|f_j - f\|_u \longrightarrow 0 \quad \text{as } j \to \infty.$$

Hence $f \in D(T_2)$. Consequently $D(T_2)$ is complete with respect to the norm $H_1((a,b))$. Therefore T_2 is closed. The operator T_2 is densely defined because $C_c^\infty((a,b)) \subset D(T_2)$.

(c) For $k = 1, \dots, 5$, the norm $\|\cdot\|_{T_k}$ is equivalent to the norm $H^1((a,b))$. Thus the same strategy used in (b) can be used to prove that T_3, T_4 and T_5 are closed.

Problem 4. We first recall two general facts. Let A and B be closed densely defined operators on H. If $A \subset B$, then $B^* \subset A^*$. Furthermore $A^{**} = A$.

We will show that $T_1^* \subset T_5$ and $T_5 \subset T_1^*$.

By the above remark, to prove that $T_1^* \subset T_5$ it suffices to prove that $T_5^* \subset T_1$. Let $f \in D(T_5^*)$ and $g \in C_c^\infty((a,b))$. We observe that $g \in D(T_5)$ because $C_c^\infty((a,b)) \subset H_0^1((a,b))$. By the definition of $D(T_5^*)$, we have

$$\langle T_5^* f, g\rangle = \langle f, T_5 g\rangle.$$

By the definition of T_5 we have

$$\langle T_5^* f, g\rangle = -i\langle f, g'\rangle.$$

Using integration by parts we obtain

$$\langle f, g'\rangle = i\langle T_5^* f, g\rangle.$$

By the definition of weak derivative, this means that $f \in H^1((a,b))$ and $f' = -iT_5^* f$. In other words $T_5^* f = if'$. Thus $T_5^* \subset T_1$. This proves that $T_1^* \subset T_5$.

Let $g \in D(T_5)$. For all $f \in D(T_1)$, we have

$$\langle T_1 f, g\rangle = -i\langle f', g\rangle = -i\langle f, g'\rangle.$$

Here we used the definitions of T_1 and T_5 and integration by parts. By the definition of T_1^* we conclude that $g \in D(T_1^*)$ and $T_1^* g = ig'$. Thus $T_5 \subset T_1^*$.

Thus we have proved that $T_1^* \subset T_5$ and $T_5 \subset T_1^*$, that is, $T_1^* = T_5$.