

Problem Set 2

Operators on Hilbert spaces

1 Problems

1. Let $(a_j)_{j \in \mathbb{N}}$ be a sequence of complex numbers. Consider the operator

$$M_a : \begin{cases} l^2 \rightarrow l^2 \\ (x_1, x_2, x_3, \dots) \mapsto (a_1 x_1, a_2 x_2, a_3 x_3, \dots). \end{cases}$$

- (a) Prove that M_a is bounded if and only if $(a_j)_{j \in \mathbb{N}}$ is bounded.
(b) Prove that if M_a is bounded then $\|M_a\| = \sup_{j \in \mathbb{N}} |a_j|$.

2. Is the operator

$$M : \begin{cases} l^2 \rightarrow l^2 \\ (x_j)_{j \in \mathbb{N}} \mapsto (\frac{1}{j} x_j)_{j \in \mathbb{N}} \end{cases}$$

bijjective? Give an example to support your answer.

3. For $k = 1, \dots, 5$, consider the operators

$$T_k : \begin{cases} D(T_k) \rightarrow L^2([a, b]) \\ f \mapsto i f' \end{cases}$$

with

$$\begin{aligned} D(T_1) &= H^1((a, b)), \\ D(T_2) &= \{f \in H^1((a, b)) \mid f(a) = 0\}, \\ D(T_3) &= \{f \in H^1((a, b)) \mid f(b) = 0\}, \\ D(T_4) &= \{f \in H^1((a, b)) \mid f(a) = f(b)\}, \\ D(T_5) &= H_0^1((a, b)) = \{f \in H^1((a, b)) \mid f(a) = f(b) = 0\}. \end{aligned}$$

In Lecture 2, we mentioned that the operators T_1, \dots, T_5 are closed with dense domains.

- (a) Prove that T_1 is closed with dense domain.
(b) Prove that T_2 is closed with dense domain.

- (c) Conclude that the strategy used to prove (b) may be used to prove that T_3 , T_4 and T_5 are closed with dense domains.

To solve this problem, you may use basic theorems of analysis and theorems mentioned in class. In particular, the following theorem may be useful:

Theorem 1 (Sobolev embedding). *The space $H^1((a, b))$ is a subset of $C([a, b])$. Furthermore, there exists a constant $C > 0$ such that*

$$\|f\|_u \leq C\|f\|_{H^1((a,b))} \quad \text{for all } f \in H^1((a, b)).$$

Here $\|f\|_u = \sup_{x \in [a,b]} |f(x)|$.

An element of $H^1((a, b))$ is an equivalence class of functions that are equal almost everywhere, and the Sobolev embedding theorem states that each such equivalence class contains a continuous function on $[a, b]$. Equivalently, the Sobolev embedding theorem states that there exists a continuous mapping (or embedding)

$$J : H^1((a, b)) \rightarrow (C([a, b]), \|\cdot\|_u)$$

that identifies a function f regarded as an element of $H^1((a, b))$ with the same function f regarded as an element of $C([a, b])$.

4. Consider the operators T_1, \dots, T_5 in Problem 3. In Lecture 2, we mentioned that $T_1^* = T_5$, $T_2^* = T_3$, $T_3^* = T_2$, $T_4^* = T_4$, and $T_5^* = T_1$. Prove that $T_1^* = T_5$.