Problem Set 3
Compact operators and spectrum

1 Problems

1. Let $(a_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Consider the operator

$$M_a : \begin{cases} l^2 \to l^2 \\ (x_1, x_2, x_3, \dots) \mapsto (a_1 x_1, a_2 x_2, a_3 x_3, \dots). \end{cases}$$

   (a) Prove that M_a is compact if and only if $\lim_{j\to\infty} a_j = 0$.

   (b) Suppose that $\sup_{j\in\mathbb{N}} |a_j| < \infty$. Prove that M_a is bijective and has a bounded inverse if and only if $\inf_{j\in\mathbb{N}} |a_j| > 0$. Give a formula for M_a^{-1}.

   (c) Find the spectrum of M_a and show that $\sigma_p(M_a) = \{a_j \mid j \in \mathbb{N}\}$.

2. For $k \in L^2([0,1]\times[0,1])$, consider the operator

$$T : \begin{cases} L^2([0,1]) \to L^2([0,1]) \\ f(\cdot) \mapsto \int_0^1 k(\cdot, y) f(y)\, dy. \end{cases}$$

   (a) Prove that $\|T\| \le \|k\|_{L^2([0,1]\times[0,1])}$.

   (b) Prove that T is compact.

   (To solve this problem you may use the following fact: If $\{u_j \mid j \in \mathbb{N}\}$ is an orthonormal basis for $L^2([0,1])$, then $\{(x,y) \mapsto u_j(x) u_k(y) \mid j,k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([0,1]\times[0,1])$.)

3. Consider the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\, dy \quad \text{for } x \in [0,1]$$

   where

$$k(x,y) = \begin{cases} 1 & \text{if } x \ge y \\ 0 & \text{if } x < y \end{cases}$$

   for $(x,y) \in [0,1]\times[0,1]$. Find the spectrum of T.