Problem Set 3
Compact operators and spectrum

1 Problems

1. Let \((a_j)_{j \in \mathbb{N}}\) be a sequence of complex numbers. Consider the operator

\[ M_a : \ell^2 \to \ell^2 \]

\[ (x_1, x_2, x_3, \ldots) \mapsto (a_1 x_1, a_2 x_2, a_3 x_3, \ldots). \]

(a) Prove that \(M_a\) is compact if and only if \(\lim_{j \to \infty} a_j = 0\).

(b) Suppose that \(\sup_{j \in \mathbb{N}} |a_j| < \infty\). Prove that \(M_a\) is bijective and has a bounded inverse if and only if \(\inf_{j \in \mathbb{N}} |a_j| > 0\). Give a formula for \(M_a^{-1}\).

(c) Find the spectrum of \(M_a\) and show that \(\sigma_p(M_a) = \{a_j \mid j \in \mathbb{N}\}\).

2. For \(k \in L^2([0,1] \times [0,1])\), consider the operator

\[ T : \begin{cases} L^2([0,1]) \to L^2([0,1]) \\ f(\cdot) \mapsto \int_0^1 k(\cdot, y) f(y) \, dy. \end{cases} \]

(a) Prove that \(\|T\| \leq \|k\|_{L^2([0,1] \times [0,1])}\).

(b) Prove that \(T\) is compact.

(To solve this problem you may use the following fact: If \(\{u_j \mid j \in \mathbb{N}\}\) is an orthonormal basis for \(L^2([0,1])\), then \(\{(x,y) \mapsto u_j(x) u_k(y) \mid j, k \in \mathbb{N}\}\) is an orthonormal basis for \(L^2([0,1] \times [0,1])\).)

3. Consider the operator \(T : L^2([0,1]) \to L^2([0,1])\) defined by

\[ (Tf)(x) = \int_0^1 k(x, y) f(y) \, dy \quad \text{for} \ x \in [0,1] \]

where

\[ k(x, y) = \begin{cases} 1 & \text{if} \ x \geq y \\ 0 & \text{if} \ x < y \end{cases} \]

for \((x,y) \in [0,1] \times [0,1]\). Find the spectrum of \(T\).