Asymptotics for Fermi curves of electric and magnetic periodic fields

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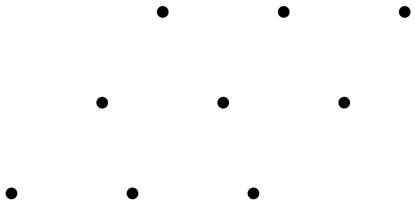
Outline

Lattice

- Γ is a lattice in $\mathbb{R}^2$:



For example $\Gamma = \mathbb{Z}^2$.

Periodic potentials

- A_1, A_2 and V are functions from $\mathbb{R}^2$ to $\mathbb{R}$
  periodic with respect to Γ.
- $A := (A_1, A_2)$ is the magnetic potential.
- V is the electric potential.

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Hamiltonian

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$$H = (i\nabla + A)^2 + V$$
acting on $L^2(\mathbb{R}^2)$, where ∇ is the gradient on $\mathbb{R}^2$.

- Spectrum of H is continuous:
$$H \text{ has no eigenfunctions in } L^2(\mathbb{R}^2).$$

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H has no eigenfunctions in $L^2(\mathbb{R}^2)$.

Translational symmetry

- But H commutes with translations:

$$HT_\gamma = T_\gamma H \qquad \text{for all } \gamma \in \Gamma,$$

where

$$T_\gamma : \varphi(x) \mapsto \varphi(x + \gamma).$$

Bloch theory

- Hence there are simultaneous eigenvectors for $\{\ H$ and T_γ for all $\gamma \in \Gamma\ \}$

$$H\,\varphi_{n,k} = E_n(k)\,\varphi_{n,k},$$

$$\varphi_{n,k}(\cdot + \gamma) = T_\gamma\,\varphi_{n,k} = e^{ik\cdot\gamma}\,\varphi_{n,k} \qquad \text{for all } \gamma \in \Gamma,$$

where $k \in \mathbb{R}^2$ and $n \in \{1, 2, 3, \dots\}$.

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- Equivalently, if we define

$$H_k := e^{-ik\cdot x} H\, e^{ik\cdot x} = (i\nabla + A - k)^2 + V,$$

we may consider the k-family of problems

$$H_k\, \psi_{n,k} = E_n(k)\, \psi_{n,k} \qquad \text{for} \qquad \psi_{n,k} \in L^2(\mathbb{R}^2/\Gamma).$$

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- The function $k \mapsto E_n(k)$ is periodic with respect to the dual lattice

$$\Gamma^{\#} := \{b \in \mathbb{R}^2 \mid b \cdot \gamma \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Gamma\}.$$

- This framework is preserved if we complexify:

$$A_1, A_2, V \in \mathbb{C} \qquad \text{and} \qquad k \in \mathbb{C}^2.$$

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Fermi curve

- The real lifted Fermi curve:

$$\widehat{\mathcal{F}}_{\lambda,\mathbb{R}} := \{k \in \mathbb{R}^2 \mid E_n(k) = \lambda \quad \text{for some } n \geq 1\}$$
$$= \{k \in \mathbb{R}^2 \mid (H_k - \lambda)\varphi = 0 \quad \text{for some } \varphi \in \mathcal{D}_{H_k} \setminus \{0\}\}.$$

- Without loss of generality

$$A - \textstyle\int A \to A, \qquad V - \lambda \to V, \qquad k \to k + \textstyle\int A.$$

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Fermi curve: properties

The Fermi curve is:

1. Analytic:

$$\widehat{\mathcal{F}} = \{k \in \mathbb{C}^2 \mid F(k) = 0\},$$

where $F(k)$ is an analytic function on $\mathbb{C}^2$.

2. Periodic with respect to $\Gamma^\#$:

$$\widehat{\mathcal{F}} + b = \widehat{\mathcal{F}} \quad \text{for all } b \in \Gamma^\#.$$

3. Gauge invariant:

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The free Hamiltonian

- Set $A = 0$ and $V = 0$. Then

$$\{e^{ib\cdot x} \mid b \in \Gamma^{\#}\}$$

is a basis of $L^2(\mathbb{R}^2/\Gamma)$ of eigenfunctions of H_k:

$$\begin{aligned} H_k\, e^{ib\cdot x} &= (i\nabla - k)^2\, e^{ib\cdot x} \\ &= (-b-k)^2\, e^{ib\cdot x} =: N_b(k)\, e^{ib\cdot x} \\ &\qquad\qquad = N_{b,1}(k)N_{b,2}(k)\, e^{ib\cdot x} \end{aligned}$$

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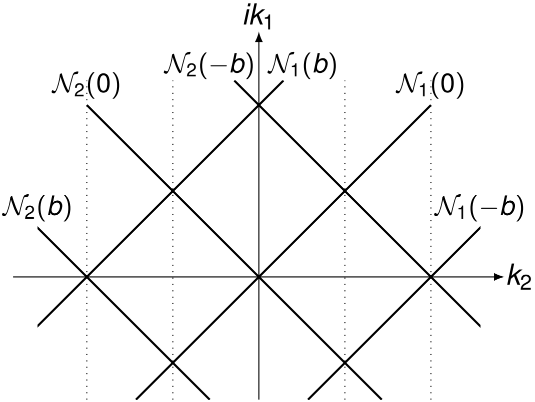
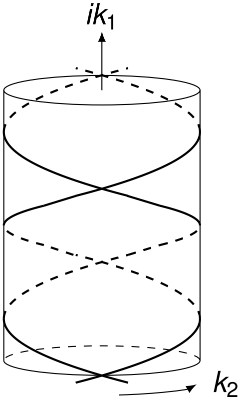
Sketch of the free Fermi curve (for ik_1 and k_2 real)



$\widehat{\mathcal{F}}$



$\mathcal{F} := \widehat{\mathcal{F}}/\Gamma^{\#}$

Main results

- Let 2Λ be the length of the shortest b in $\Gamma^{\#}$.
- Fix $\varepsilon < \Lambda/6$.
- Assume that A and V "are differentiable".
- Notation: $\widehat{\mathcal{F}} \equiv \widehat{\mathcal{F}}(A, V)$

"Theorem".
Suppose that $\|A\|_{L^2} \lesssim \varepsilon$ (small).
Then, outside of a compact set (asymptotically),
the curve $\widehat{\mathcal{F}}(A, V)$ is very close to $\widehat{\mathcal{F}}(0, 0)$,
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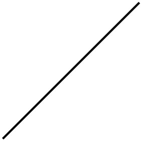

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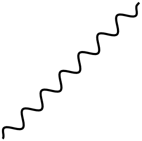

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$z_1 z_2 = 0$

can open up to

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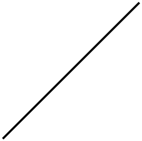

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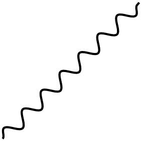

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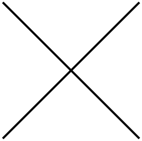

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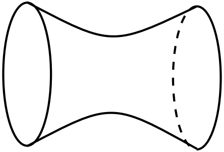

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Remarks

- Generically all double points open up:
  1-D complex manifold.
- For $A = 0$ proved by
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Idea of proof

- Write

$$\begin{aligned} H_k &= (i\nabla + A - k)^2 + V \\ &= (i\nabla - k)^2 + 2A \cdot (i\nabla - k) + q \end{aligned}$$

where

$$q := (i\nabla \cdot A) + A^2 + V.$$

Idea of proof

- Then $k \in \widehat{\mathcal{F}}(A,V)$ if and only if

$$\Big[(i\nabla - k)^2 + 2A\cdot(i\nabla - k) + q\Big]\varphi = 0$$

for $\varphi \in L^2(\mathbb{R}^2/\Gamma)$ with $\varphi \neq 0$, or, equivalently,

$$\Big[N_c(k)\delta_{b,c} - 2(c+k)\cdot\hat{A}(b-c) + \hat{q}(b-c)\Big]_{b,c\in\Gamma^\#}\begin{bmatrix} | \\ \hat{\varphi}(c) \\ | \end{bmatrix}_{c\in\Gamma^\#} = 0,$$

where $\hat{f}(b) = \int_{\mathbb{R}^2/\Gamma} f(x)\, e^{-ib\cdot x}\, dx$.

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- ε-tubes about $\mathcal{N}_b$:

$$T_b := T_1(b) \cup T_2(b),$$
$$T_\nu(b) := \{k \in \mathbb{C}^2 \mid \quad |N_{b,\nu}(k)| < \varepsilon\}.$$

- Write

$$k = u + iv \qquad \text{with} \qquad u, v \in \mathbb{R}^2.$$

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- Let $G = \{0\}$ or $G = \{0, d\}$ with $0, d \in \Gamma^\#$.
  We can split our equation:

$$\Big[N_c(k)\delta_{b,c} - 2(c+k)\cdot\hat{A}(b-c) + \hat{q}(b-c)\Big]_{\substack{b\in G\\ c\in\Gamma^\#}} \begin{bmatrix} | \\ \hat{\varphi}(c) \\ | \end{bmatrix}_{c\in\Gamma^\#} = 0,$$

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$$= -\Big[-2(c+k)\cdot\hat{A}(b-c) + \hat{q}(b-c)\Big]_{\substack{b\in\Gamma^\#\setminus G \\ c\in G}}\begin{bmatrix} | \\ \hat{\varphi}(c) \\ | \end{bmatrix}_{c\in G}$$

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- Rewriting again the second equation:

$$\underbrace{\left[\delta_{b,c} - \frac{2(c+k)}{N_c(k)} \cdot \hat{A}(b-c) + \frac{\hat{q}(b-c)}{N_c(k)}\right]_{b,c\in\Gamma^{\#}\setminus G}}_{=:R_{G'G'}} \begin{bmatrix} | \\ N_c(k)\hat{\varphi}(c) \\ | \end{bmatrix}_{c\in\Gamma^{\#}\setminus G}$$

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Idea of proof

- Substituting in the first equation
  we conclude that it has a solution if and only if

$$\det\left[N_{d'}(k)\delta_{d',d''}+w_{d',d''}-\sum_{b,c\in G'}\frac{w_{d',b}}{N_b(k)}(R^{-1}_{G'G'})_{b,c}w_{c,d''}\right]_{d',d''\in G}=0,$$

where

$$w_{b,c}:=-2(c+k)\cdot\hat{A}(b-c)+\hat{q}(b-c).$$

This is a $|G|\times|G|$ determinant.

Idea of proof

Hence we have local defining equations for $\widehat{\mathcal{F}}(A, V)$:

- Deformed planes ($G = \{0\}$):

$$N_0(k) + D_{00}(k) = 0.$$

- Handles ($G = \{0, d\}$):

$$(N_0(k) + D_{00}(k))(N_d(k) + D_{dd}(k)) = D_{0,d} D_{d,0}.$$

where

$$D_{d',d''}(k) := B_{11}^{d'd''} k_1^2 + B_{22}^{d'd''} k_2^2 + (B_{12}^{d'd''} + B_{21}^{d'd''}) k_1 k_2 + C_1^{d'd''} k_1 + C_2^{d'd''} k_2 + C_0^{d'd''}.$$

Idea of proof

- Linear change of variables:

$$(k_1, k_2) \mapsto (w, z),$$

  where w is "small" and z is "large".

- Asymptotics for the coefficients:

$$\Phi_{d',d''}(k) := \sum_{b,c \in G'} \frac{f(d'-b)}{N_b(k)} (R^{-1}_{G'G'})_{b,c}\, g(c-d'')$$

$$= O(1) + O\left(\frac{1}{z}\right) + O\left(\frac{1}{z^2}\right).$$

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- **Proof:** Chain rule; Leibniz rule; $\frac{1}{1-X} = 1 + X + X^2 + \cdots$
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