Quantitative derivation of the Gross-Pitaevskii equation

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This talk is about

- Mathematics of many-body quantum mechanics.
- Dynamics of Bose-Einstein condensates.
- Effective description.
- How the Gross-Pitaevskii PDE emerges.

Wave function for N bosons

- N-particle wave function:

$$\psi_N(x_1, \ldots, x_N, t) \in \mathbb{C}, \qquad x_1, \ldots, x_N \in \mathbb{R}^3, \qquad t \in \mathbb{R}.$$

- Square-integrable and normalized:

$$\psi_N(\cdot\,, t) \in L^2(\mathbb{R}^{3N}) \qquad \text{and} \qquad \int_{\mathbb{R}^{3N}} |\psi_N(\cdot\,, t)|^2 = 1.$$

- ψ_N is symmetric in each pair of variables $x_1, \ldots, x_N$.

Density operator

$$|\psi_N\rangle\langle\psi_N| \quad \text{on} \quad L^2(\mathbb{R}^{3N}) \simeq L^2(\mathbb{R}^3) \otimes \cdots \otimes L^2(\mathbb{R}^3).$$

Bose-Einstein condensate

In experiments (since 1995)

Trapped cold ($T \sim 10^{-9}K$) dilute gas of $N \sim 10^3$ bosons.

Heuristically

$$\psi_N(x_1, \ldots, x_N, t_0) \simeq \prod_{j=1}^{N} \varphi(x_j) \quad \text{where} \quad \varphi \in L^2(\mathbb{R}^3).$$

$$|\psi_N\rangle\langle\psi_N| \simeq |\varphi\rangle\langle\varphi| \otimes \cdots \otimes |\varphi\rangle\langle\varphi|.$$

Condensate states

One-particle reduced density operator

$$|\psi_N\rangle\langle\psi_N|^{(1)} = \mathrm{Trace}_{2\to N}|\psi_N\rangle\langle\psi_N|.$$

(Integrate out $N-1$ variables of the integral kernel of $|\psi_N\rangle\langle\psi_N|$.)

$|\psi_N\rangle\langle\psi_N|^{(1)}$ plays the role of one-particle wave-function.

Condensate condition

$$\mathrm{Tr}\,\Big|\, |\psi_N\rangle\langle\psi_N|^{(1)} - |\varphi\rangle\langle\varphi| \,\Big| \leq \frac{C}{N} \to 0 \quad \text{as} \quad N\to\infty.$$

Model (which is realistic)

Quantum Hamiltonian in the Gross-Pitaevskii regime

$$H_N^{\mathrm{trap}} = \sum_{j=1}^{N} \big(-\Delta_{x_j} + V_{\mathrm{trap}}(x_j)\big) + \frac{1}{N}\sum_{i<j}^{N} N^3 V(N(x_i - x_j)),$$

$$V_{\mathrm{trap}}(y) = |y|^2 \qquad \text{and} \qquad V \geq 0 \text{ with compact support.}$$

Very heuristically

$$\frac{1}{N} N^3 V(N\,\cdot\,) \sim \frac{1}{N}\delta(\cdot) \qquad \text{for large } N$$

models rare but strong collisions.

Time evolution of condensates

Initial condition

$\psi_N|_{t=0} = \theta_N =$ condensate state with correlations (not a product)

We construct initial data Θ in Fock space:

$$\Theta = W(\sqrt{N}\varphi)T(k)\Omega = \theta_0 \oplus \theta_1 \oplus \cdots \oplus \theta_N \oplus \cdots \in \bigoplus_{n\geq 0} L^2_{sym}(\mathbb{R}^{3n})$$

$$\begin{aligned}
\Omega &= \text{finite particle state (e.g. vacuum)} \\
T(k) &= \text{Bogoliubov transformation} \\
k(x,y) &= \text{integral kernel which models correlations} \\
W(\sqrt{N}\varphi) &= \text{Weyl operator} \\
\varphi(x) &= \text{one particle state} \\
\Theta &= \text{modified coherent state}
\end{aligned}$$

Schrödinger equation on Fock space

Condensate state reached; traps are turned off

$$H_N = H_N^{\mathrm{trap}} \text{ with } V_{\mathrm{trap}} \equiv 0.$$

Hamiltonian on Fock space

$$\mathcal{H} = H_0 \oplus H_1 \oplus \cdots \oplus H_N \oplus \cdots$$

Time evolution is observed

$$\begin{cases} i\partial_t \Psi = \mathcal{H}\Psi \\ \Psi|_{t=0} = W(\sqrt{N}\varphi)T(k)\Omega \end{cases} \quad \text{as} \quad N \to \infty.$$

Theorem [Benedikter, deO, Schlein, CPAM 2014]

Reasonable hypothesis on $V \geq 0$, φ, Ω.
Consider the solution

$$\Psi = e^{-i\mathcal{H}t} W(\sqrt{N}\varphi) T(k) \Omega.$$

Let

$$\Gamma^{(1)}_{N,t} = \text{one-particle reduced operator of } \Psi.$$

Then

$$\mathrm{Tr} \left| \Gamma^{(1)}_{N,t} - |\varphi\rangle\langle\varphi| \right| \leq C \exp(C \exp(C|t|)) \frac{1}{\sqrt{N}}$$

for all t and N, where φ_t solves (time-dep. Gross-Pitaevskii eqn.)

$$i\partial_t \varphi_t = -\Delta \varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t \qquad \text{with} \qquad \varphi_t|_{t=0} = \varphi,$$

$a_0 > 0$ (scattering length of V).

Remarks

Based on

- Hepp '74, Ginibre–Velo '79, Rodnianski–Schlein '09,. . .

Previous results

- Spohn '80, Erdös–Schlein–Yau '06, Pickl '10,. . .
  (no rate of convergence)

Other results

- Adami–Golse–Teta '07, Grillakis–Machedon–Margetis '10,. . .

Large bibliography. . .

Look at arXiv:1208.0373 and Schlein's notes arXiv:1210.1603.

Thank you for your attention!