Vector Spaces

Contents

1 Axioms of vector space

Let $\mathbb{F}$ be the field of real numbers or the field of complex numbers:

$$\mathbb{F} = \mathbb{R} \qquad \text{or} \qquad \mathbb{F} = \mathbb{C}.$$

A **vector space** over the field $\mathbb{F}$ is a set X together with an operation of addition of elements of X, denoted by

$$\begin{aligned} X \times X &\to X \\ (x, y) &\mapsto x + y, \end{aligned}$$

and an operation of multiplication of elements of X by numbers in $\mathbb{F}$, denoted by

$$\begin{aligned} \mathbb{F} \times X &\to X \\ (\alpha, x) &\mapsto \alpha x, \end{aligned}$$

with the following properties. For all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{F}$:

- The addition is commutative and associative, that is,

$$x + y = y + x, \tag{V1}$$

$$(x + y) + z = x + (y + z). \tag{V2}$$

- There exists an element $0 \in X$, called the element **zero**, such that

$$x + 0 = x \quad \text{for all } x \in X. \tag{V3}$$

- For each $x \in X$, there exists $-x \in X$, called the **opposite** of x, such that

$$x + (-x) = 0. \tag{V4}$$

- The multiplication by numbers obeys

$$(\alpha\beta)x = \alpha(\beta x), \tag{V5}$$

$$1x = x. \tag{V6}$$

- The addition and the multiplication by numbers satisfy the distributive properties

$$\alpha(x + y) = \alpha x + \alpha y, \tag{V7}$$

$$(\alpha + \beta)x = \alpha x + \beta x. \tag{V8}$$

A vector space is called a **real vector space** if $\mathbb{F} = \mathbb{R}$ and a **complex vector space** if $\mathbb{F} = \mathbb{C}$. Elements of a vector space are called **vectors** and elements of $\mathbb{F}$ are called **scalars** (or **numbers**). The operations defined on a vector space behave like the operations with vectors in the plane (more on this later). The essential features of these operations are abstracted in the Axioms V1-V8.

Remark 1. We denote the vector zero and the number zero by the same symbol 0 and the operations of addition of vectors and addition of numbers by the same symbol $+$.

2 Examples of vector spaces

We now list some examples of vector spaces. To specify a vector space, we should specify a set, the vector space operations defined on the set, and we must verify that the Axioms V1-V8 are satisfied.

Example 1 (Arrows in a plane)**.** In a plane, consider the set of arrows starting at a point P. These arrows are used in physics to describe forces or velocities at the point P. We can define the operations of addition of arrows and multiplication of an arrow by a number in the usual way. These operations satisfy the Axioms V1-V8.

Example 2 (Space $\mathbb{F}[z]$ of polynomials)**.** A polynomial in the variable z with coefficients in $\mathbb{F}$ and degree k is an expression of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k,$$

where $a_k \neq 0$. Here, k is a non-negative integer, called the degree of the polynomial, and $a_0, a_1, \ldots, a_k$ are numbers in $\mathbb{F}$, called the coefficients. By definition, we have $z^0 = 1$ and $z^1 = z$. For $j = 0, \dots, k$, the number a_j is called the coefficient of the monomial $a_j z^j$. The number a_0 is called the constant term. We denote by $\mathbb{F}[z]$ the set of all polynomials in z with coefficients in $\mathbb{F}$. Let p and q be polynomials in $\mathbb{F}[z]$ and let α be a number in $\mathbb{F}$. The equality $p = q$ of polynomials means that p and q have the same degree and the same corresponding monomial coefficients. The polynomial $p+q$ is defined by adding the corresponding monomial coefficients of p and q. The polynomial αp is defined by multiplying all the coefficients of p by α. The polynomial zero, denoted by 0, has only the constant term which is equal to zero, and the polynomial $-p$ is obtained by multiplying all the coefficients of p by -1. The operations obtained in this way satisfy the Axioms V1-V8. Thus $\mathbb{F}[z]$ is a vector space.

Example 3 (Space $\mathbb{F}^n$ of n-tuples)**.** We denote by $\mathbb{F}^n$ the set of all ordered n-tuples of numbers in $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{F}\}.$$

It is customary to use the notation

$$x = (x_1, \dots, x_n)$$

to represent a tuple in $\mathbb{F}^n$. For $j = 1, \dots, n$, the number x_j is called the jth coordinate of x. Let x and y be tuples in $\mathbb{F}^n$ and let α be a number in $\mathbb{F}$. The equality $x = y$ of tuples means that $x_j = y_j$ for all $j = 1, \dots, n$. The operation of addition of tuples is defined by

$$x + y = (x_1 + y_1, \dots, x_n + y_n).$$

The operation of multiplication of a tuple by a number is defined by

$$\alpha x = (\alpha x_1, \dots, \alpha x_n).$$

The tuple zero is given by $0 = (0, \dots, 0)$, and the opposite of x is given by $-x = (-x_1, \dots, -x_n)$. Using the properties of the addition and the multiplication of numbers in $\mathbb{F}$, it is simple to verify that $\mathbb{F}^n$ with the above operations is a vector space. The space $\mathbb{R}^n$ is called the **real coordinate n-space**. The space $\mathbb{C}^n$ is called the **complex coordinate n-space**.

Example 4 (Space $M(m \times n, \mathbb{F})$ of matrices)**.** An $m \times n$ matrix of numbers is a collection of mn numbers indexed by a pair (i, j) where $i = 1, \dots, m$ and $j = 1, \dots, n$. It is customary to denote a matrix by

$$A = [a_{ij}] \qquad \text{or} \qquad A = [a_{ij}]_{m \times n}.$$

The number a_{ij} is called the (i, j) entry of $[a_{ij}]$. To refer to the (i, j) entry of A, we use the notation $[A]_{ij}$. We denote by $M(m \times n, \mathbb{F})$ the set of all $m \times n$ matrices with entries in $\mathbb{F}$:

$$M(m \times n, \mathbb{F}) = \{[a_{ij}]_{m \times n} \mid a_{ij} \in \mathbb{F} \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, n\}.$$

If $m = n$, we write $M(n, \mathbb{F})$ in place of $M(n \times n, \mathbb{F})$. We usually represent a matrix $[a_{ij}]$ as a table with m rows and n columns surrounded by brackets or parenthesis, in which the number a_{ij} is in the ith row and jth column:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Let A and B be matrices in $M(m \times n, \mathbb{F})$ and let α be a number in $\mathbb{F}$. The equality $A = B$ of matrices means that $a_{ij} = b_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. The operation of addition of matrices is defined by

$$A + B = [a_{ij} + b_{ij}].$$

The operation of multiplication of a matrix by a number is defined by

$$\alpha A = [\alpha a_{ij}].$$

The matrix zero is given by $0 = [0]$, and the opposite of A is given by $-A = [-a_{ij}]$. The set $M(m \times n, \mathbb{F})$ with the above operations is a vector space.

Example 5 (Space $\mathbb{F}^{\mathbb{N}}$ of sequences)**.** We denoted by $\mathbb{F}^{\mathbb{N}}$ the set of all sequences of numbers in $\mathbb{F}$:

$$\mathbb{F}^{\mathbb{N}} = \{(x_j)_{j=1}^{\infty} \mid x_j \in \mathbb{F} \text{ for } j \in \mathbb{N}\}.$$

It is customary to use the notation

$$x = (x_j)_{j=1}^{\infty}$$

to represent a sequence in $\mathbb{F}^{\mathbb{N}}$. Let x and y be sequences in $\mathbb{F}^{\mathbb{N}}$ and let α be a number in $\mathbb{F}$. The equality $x = y$ of sequences means that $x_j = y_j$ for all $j \in \mathbb{N}$. The number x_j is called the jth term of x. The operation of addition of sequences is defined by

$$x + y = (x_j + y_j)_{j=1}^{\infty}.$$

The operation of multiplication of a sequence by a number is defined by

$$\alpha x = (\alpha x_j)_{j=1}^{\infty}.$$

The sequence zero is given by $0 = (0)_{j=1}^{\infty}$, and the opposite of x is given by $-x = (-x_j)_{j=1}^{\infty}$. The set $\mathbb{F}^{\mathbb{N}}$ with the above operations is a vector space.

Example 6 (Space $\mathbb{F}^{[a,b]}$ of functions)**.** Let $[a, b]$ be an interval of $\mathbb{R}$. We denote by $\mathbb{F}^{[a,b]}$ the set of all $\mathbb{F}$-valued functions on $[a, b]$. Let f and g be functions in $\mathbb{F}^{[a,b]}$ and let α be a number in $\mathbb{F}$. The equality $f = g$ of functions means that $f(x) = g(x)$ for all $x \in [a, b]$. The operation of addition of functions is defined by

$$(f + g)(x) = f(x) + g(x).$$

The operation of multiplication of a function by a number is defined by

$$(\alpha f)(x) = \alpha f(x).$$

The function zero, denoted by 0, is given by $0(x) \equiv 0$. The opposite of f, denoted by $-f$, is given by $(-f)(x) = -f(x)$. The set $\mathbb{F}^{[a,b]}$ with the above operations is a vector space.

Example 7 (Space Y^X of mappings and the pointwise principle)**.** Let X be a set and let Y be a vector space. We denote by Y^X the set of all mappings $f : X \to Y$. Let f and g be mappings in Y^X and let α be a number in $\mathbb{F}$. We now present the so-called pointwise principle, which consists of the following statements.

Pointwise principle:

- The equality $f = g$ of mappings means that

$$f(x) = g(x) \quad \text{for all } x \in X.$$

- The sum $f + g$ of mappings is defined by

$$(f + g)(x) = f(x) + g(x).$$

- The product αf of a mapping by a number is defined by

$$(\alpha f)(x) = \alpha f(x).$$

- The mapping zero, denoted by 0, is given by

$$0(x) \equiv 0.$$

- The opposite of f, denoted by $-f$, is given by

$$(-f)(x) = -f(x).$$

The set Y^X together with the pointwise principle is a vector space. By varying the set X and the vector space Y we obtain many examples of vector spaces. For instance

$$\begin{aligned} \mathbb{F}^{\{1,\ldots,n\}} &= \mathbb{F}^n, \\ \mathbb{F}^{\{1,\ldots,m\}\times\{1,\ldots,n\}} &= M(m \times n, \mathbb{F}), \\ \mathbb{F}^{\mathbb{N}}&, \\ \mathbb{F}^{[a,b]}&. \end{aligned}$$

The pointwise principle may be stated in the following way: If a certain operation can be performed on the values of mappings at each point, then that operation is defined on the mappings themselves by simply performing the operation on the values of the mappings at each point.

Therefore, spaces of n-tuples, spaces of matrices, and spaces of functions are vector spaces. Other vector spaces arise as subsets of vector spaces when we impose some additional condition. For example:

1. The set of solutions of a system of homogeneous linear equations is a subset of $\mathbb{F}^n$ and is also a vector space.

2. The set of sequences that have only a finite number of nonzero terms is a subset of $\mathbb{F}^{\mathbb{N}}$ and is also a vector space.

3. The set of continuous functions from $[a, b]$ to $\mathbb{F}$ is a subset of $\mathbb{F}^{[a,b]}$ and is also a vector space.

Subsets of vector spaces that are also vector spaces are called subspaces (more on this later).

Exercises

1. Prove that Y^X together with the pointwise principle is a vector space (see Example 7).

2. Consider the set of all positive real numbers $\mathbb{R}_+$ with addition defined by $x \oplus y = xy$ and multiplication by scalar $\alpha \in \mathbb{R}$ defined by $\alpha \odot x = x^\alpha$. Is $\mathbb{R}_+$ with $\oplus$ and $\odot$ a vector space over $\mathbb{R}$? Explain.

3. Consider the set $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. For $x, y \in \mathbb{R}$, define, in the usual way, the sum of x and y and the multiplication of x by a scalar $\alpha \in \mathbb{R}$.

In addition, define

$$x(-\infty) = \begin{cases} \infty & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -\infty & \text{if } x > 0 \end{cases}, \qquad x\infty = \begin{cases} -\infty & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \infty & \text{if } x > 0 \end{cases},$$

$$x + (-\infty) = -\infty + x = -\infty, \qquad x + \infty = \infty + x = \infty,$$

$$-\infty + (-\infty) = -\infty, \quad \infty + \infty = \infty, \quad -\infty + \infty = \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ a vector space over $\mathbb{R}$? Explain.

3 Properties of vector spaces

The axioms of vector space imply that the usual rules of vector algebra and properties like the uniqueness of the vector zero and of the opposite hold true in a vector space. These properties may be obvious for the space $\mathbb{F}^n$, but they should be derived from the Axioms V1-V8. We now present some of these properties.

Let x, y and z be elements of a vector space over $\mathbb{F}$, and let α be a number in $\mathbb{F}$. We have the following properties:

(a) There exists a unique vector zero. In fact, if 0 and $0'$ are vectors zero, then

$$0 \overset{V3}{=} 0 + 0' \overset{V1}{=} 0' + 0 \overset{V3}{=} 0'.$$

(b) For each x, there exists a unique $-x$. In fact, if $y = -x$ and $z = (-x)'$ are opposites of x, that is, $x + y = 0$ and $x + z = 0$, then

$$y \overset{V3}{=} y + 0 = y + (x + z) \overset{V2}{=} (y + x) + z \overset{V1}{=} (x + y) + z = 0 + z \overset{V1}{=} z + 0 \overset{V3}{=} z.$$

(c) $0x = 0$.

(d) $\alpha 0 = 0$.

(e) $(-1)x = -x$.

(f) If $\alpha x = 0$, then $\alpha = 0$ or $x = 0$.

(g) If $x + y = x + z$, then $y = z$.

As the above properties show, we can operate with vectors in a vector space in the same way that we operate with vectors in the plane.

Because of the above properties, we may write

$$\begin{aligned} x - y &\text{ in place of } x + (-y), \\ -\alpha x &\text{ in place of } (-\alpha)x, \\ x + y + z &\text{ in place of } (x+y)+z \text{ or } x+(y+z), \\ \alpha\beta x &\text{ in place of } (\alpha\beta)x \text{ or } \alpha(\beta x). \end{aligned}$$

Therefore, if $x_1, x_2, \ldots, x_k$ are vectors and $a_1, a_2, \ldots, a_k$ are numbers, we can unambiguously form the vector

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k.$$

Exercises

1. Prove the Properties (c)-(g) on Page 7.

4 Linear combinations

Let X be a vector space over $\mathbb{F}$. For $k \in \mathbb{N}$, a **linear combination** of k vectors $x_1, \ldots, x_k \in X$ is a vector of the form

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k,$$

where $a_1, \ldots, a_k \in \mathbb{F}$. The scalars $a_1, \ldots, a_k$ are called the coefficients of the linear combination. The linear combination is said to be **trivial** if $a_j = 0$ for all $j = 1, \ldots, k$. If at least one of the coefficients $a_1, \ldots, a_k$ is different from zero, the linear combination is said to be nontrivial.

Using the axioms of vector space, it is simple to prove the following formulae (where $b_1, \ldots, b_k \in \mathbb{F}$):

$$\begin{aligned} \sum_{j=1}^{k} a_jx_j + \sum_{j=1}^{k} b_jx_j &= \sum_{j=1}^{k}(a_j + b_j)x_j, \\ \alpha\sum_{j=1}^{k} a_jx_j &= \sum_{j=1}^{k}(\alpha a_j)x_j. \end{aligned}$$

Therefore, we can informally restate the definition of vector space as follows: A vector space is a set in which we can form linear combinations in an associative, commutative, and distributive way.