Vector Subspaces

Contents

1 Vector subspaces

A subset E of a vector space X is called a **subspace** of X (or a vector subspace of X) if E (with the operations inherited from X) is a vector space. To put it simply, a subspace is a subset of a vector space that is also a vector space.

Theorem 1. *A subset E of a vector space X is a subspace of X if and only if the following conditions hold:*

1. $0 \in E$.
2. *If $x, y \in E$, then $x + y \in E$.*
3. *If $x \in E$ and $\alpha \in \mathbb{F}$, then $\alpha x \in E$.*

In other words, a subspace of a vector space is a subset that contains zero and is closed under the operations of addition and multiplication by numbers.

Proof. ($\Rightarrow$) If E is a subspace of X, then E is a vector space, and the Conditions 1-3 are satisfied.

($\Leftarrow$) Conversely, suppose that E is a subset of X and that the Conditions 1-3 are satisfied. The Condition 1 insures that the additive identity of X is in E. The Condition 2 insures that the addition is a mapping from E into E. The Condition 3 insures that the multiplication by numbers is a mapping from $\mathbb{F} \times E$ into E. Therefore, the codomain of each vector space operation is the set E and the Axioms of Vector Space V1-V3 and V5-V8 are automatically satisfied. To conclude the proof, we will show that every element of E has an additive inverse in E. In other words, we will verify the Axiom V4. If $x \in E$, then $x \in X$. Thus there exists $-x \in X$ and $-x = (-1)x$, by the properties of vector space. But $(-1)x \in E$ because of the Condition V3. Therefore $-x \in E$. □

2 Examples of subspaces

We now give some examples of subspaces. To prove that the examples are subspaces, we should verify that the Conditions 1-3 from Theorem 1 are satisfied in each case.

Example 1 (Trivial subspaces)**.** Let X be a vector space. The subset X and the subset $\{0\}$ are trivial examples of subspaces of X. The subspace $\{0\}$ is called the **null subspace**.

Example 2 (Space $\mathbb{F}_n[z]$ of polynomials)**.** We denote by $\mathbb{F}_n[z]$ the set of all polynomials in z with coefficients in $\mathbb{F}$ and degree less than or equal to n. The set $\mathbb{F}_n[z]$ is a subspace of $\mathbb{F}[z]$.

Example 3 (Line through the origin)**.** Let X be a vector space and let $v \in X$ with $v \neq 0$. The set $E = \{\alpha v \in X \mid \alpha \in \mathbb{F}\}$ is a subspace of X, called the line through the origin in the direction of v.

Example 4 (xy-plane in the real 3-space)**.** The set

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$

is a subspace of $\mathbb{R}^3$. Geometrically, we may think of E as the horizontal plane in $\mathbb{R}^3$ that crosses the origin.

Example 5 (Hyperplane of $\mathbb{F}^n$ through the origin—or the set of solutions to a homogeneous linear equation)**.** A homogeneous linear equation in the variables $x_1, \dots, x_n$ is an equation of the form

$$a_1 x_1 + \cdots + a_n x_n = 0$$

where $a_1, \dots, a_n \in \mathbb{F}$. A solution to this equation is an n-tuple of numbers $(s_1, \dots, s_n) \in \mathbb{F}^n$ such that

$$a_1 s_1 + \cdots + a_n s_n = 0.$$

Let E be the set of all the solutions to the above equation:

$$E = \{(x_1, \dots, x_n) \in \mathbb{F}^n \mid a_1 x_1 + \cdots + a_n x_n = 0\}.$$

The set E is a subspace of $\mathbb{F}^n$. In the trivial case where $a_j = 0$ for all $j = 1, \dots, n$, we have $E = \mathbb{F}^n$. If $a_j \neq 0$ for some $j \in \{1, \dots, n\}$, then E is called a hyperplane of $\mathbb{F}^n$ through the origin.

Example 6 (Space $\mathbb{F}_c^{\mathbb{N}}$ of sequences)**.** Let $\mathbb{F}_c^{\mathbb{N}}$ be the set of all sequences of numbers in $\mathbb{F}$ that have only a finite number of nonzero terms:

$$\mathbb{F}_c^{\mathbb{N}} = \{(x_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}} \mid x_j = 0 \text{ for all } j > N \text{ for some } N \in \mathbb{N}\}.$$

The set $\mathbb{F}_c^{\mathbb{N}}$ is a subspace of $\mathbb{F}^{\mathbb{N}}$.

Example 7 (Space $\mathcal{P}([a,b],\mathbb{F})$ of functions)**.** A function $p : [a,b] \to \mathbb{F}$ is called a polynomial function if there exist $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

for all $z \in [a,b]$. We denote by $\mathcal{P}([a,b],\mathbb{F})$ the set of all polynomial functions from $[a,b]$ to $\mathbb{F}$:

$$\mathcal{P}([a,b],\mathbb{F}) = \{f \in \mathbb{F}^{[a,b]} \mid f \text{ is polynomial}\}.$$

Clearly 0 is a polynomial function, and if f and g are polynomial, then $f+g$ and αf are polynomials. Thus $\mathcal{P}([a,b],\mathbb{F})$ is a subspace of $\mathbb{F}^{[a,b]}$.

Example 8 (Space $\mathcal{C}([a,b],\mathbb{F})$ of functions)**.** Let $\mathcal{C}([a,b],\mathbb{F})$ be the set of all continuous functions from $[a,b]$ to $\mathbb{F}$:

$$\mathcal{C}([a,b],\mathbb{F}) = \{f \in \mathbb{F}^{[a,b]} \mid f \text{ is continuous}\}.$$

Clearly 0 is a continuous function, and if f and g are continuous functions, then $f+g$ and αf are continuous, by a theorem of calculus. Thus $\mathcal{C}([a,b],\mathbb{F})$ is a subspace of $\mathbb{F}^{[a,b]}$. In fact, we have the following hierarchy of subspaces, one contained into the other:

$$\mathcal{P}([a,b],\mathbb{F}) \subset \mathcal{C}([a,b],\mathbb{F}) \subset \mathbb{F}^{[a,b]}.$$

Example 9. Let E be the set of all functions in $\mathbb{R}^{\mathbb{R}}$ that are solutions of the differential equation

$$u'' + u = 0.$$

The set E is a subspace of $\mathbb{R}^{\mathbb{R}}$.

3 Intersection of subspaces

The next theorem describes the behaviour of subspaces with respect to intersections.

Theorem 2 (The intersection of subspaces is a subspace)**.** *Let X be a vector space. The intersection of any family of subspaces of X is a subspace of X.*

Proof. Let $\mathcal{E}$ be a family of subspaces of X and let $Y = \bigcap_{E\in\mathcal{E}} E$. Recall that $Y = \{x \mid x \in E \text{ for all } E \in \mathcal{E}\}$. Since each $E \in \mathcal{E}$ is a subspace of X, we have $0 \in E$ for all $E \in \mathcal{E}$. Thus $0 \in Y$. Let $x, y \in Y$ and let $\alpha \in \mathbb{F}$. By the definition of Y, we have $x, y \in E$ for all $E \in \mathcal{E}$. Since each $E \in \mathcal{E}$ is a subspace of X, we have $x + y \in E$ and $\alpha x \in E$ for all $E \in \mathcal{E}$. Therefore $x + y \in Y$ and $\alpha x \in Y$. Thus Y is a subspace of X, by Theorem 1. □

Example 10 (Intersection of hyperplanes of $\mathbb{F}^n$ through the origin—or the set of solutions to a system of homogeneous linear equations)**.** A system of homogeneous linear equations in the variables $x_1, \dots, x_n \in \mathbb{F}$ is a system of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{1}$$

where $a_{ij} \in \mathbb{F}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. A solution to this system is an n-tuple of numbers $(s_1, \dots, s_n) \in \mathbb{F}^n$ that is a solution to each equation in (1). Let E be the set of all the solutions to (1). It follows that E is a subspace of $\mathbb{F}^n$. In fact, $E = E_1 \cap \cdots \cap E_m$, where E_i, for $i = 1, \dots, m$, is the set of solutions to the equation $a_{i1}x_1 + \cdots + a_{in}x_n = 0$, as in the Example 5. Thus each E_i is a subspace of $\mathbb{F}^n$. Hence E is an intersection of subspaces of $\mathbb{F}^n$. Therefore E is a subspace of $\mathbb{F}^n$, by Theorem 2.

4 Sum of subspaces

When working with vector spaces, we are usually interested in subspaces as apposed to arbitrary sets. In general, the union of subspaces is not a subspace. However, the sum of subspaces, which we describe next, is a subspace.

Let X be a vector space over $\mathbb{F}$ and let $E_1, \dots, E_k$ be subsets of X. The **sum** of $E_1, \dots, E_k$, denoted $E_1 + \cdots + E_k$, is the set of all possible sums of elements of $E_1, \dots, E_k$:

$$E_1 + \cdots + E_k = \{x_1 + \cdots + x_k \in X \mid x_1 \in E_1, \dots, x_k \in E_k\}.$$

Example 11. Suppose that

$$E_1 = \{(x, 0, 0) \in \mathbb{R}^3 \mid x \in \mathbb{R}\} \quad \text{and} \quad E_2 = \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}.$$

Then

$$E_1 + E_2 = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{F}\}.$$

Example 12. Suppose that

$$E_1 = \{(w, w, y, y) \in \mathbb{R}^4 \mid w, y \in \mathbb{R}\},$$
$$E_2 = \{(w, w, w, z) \in \mathbb{R}^3 \mid w, z \in \mathbb{R}\}.$$

Then

$$E_2 + E_2 = \{(w, w, y, z) \in \mathbb{R}^3 \mid w, x, y \in \mathbb{F}\}.$$

Theorem 3. *Suppose that $E_1, \ldots, E_k$ are subspaces of X. Then $E_1 + \cdots + E_k$ is a subspace of X. In fact, the sum $E_1 + \cdots + E_k$ is the smallest subspace of X containing $E_1, \ldots, E_k$.*

Proof. It is evident that $E_1 + \cdots + E_k$ contains the vector zero and is closed under the operations of addition and scalar multiplication. Thus $E_1 + \cdots + E_k$ is a subspace of X, by Theorem 1.

Since $E_1 + \cdots + E_k$ contains the vector $0 + 0 + \cdots + 0 + x_j + 0 + \cdots + 0$ for all $x_j \in E_j$, the subspace $E_1 + \cdots + E_k$ contains each $E_1, \ldots, E_k$. On the other hand, every subspace containing $E_1, \ldots, E_k$ also contains $E_1 + \cdots + E_k$ because every subspace contains all finite sum of their elements. Therefore, for $j = 1, \ldots, k$, we have $E_j \subset E_1 + \cdots + E_k \subset S$ for every subspace S containing $E_1, \ldots, E_k$. Thus the sum $E_1 + \cdots + E_k$ is the smallest subspace of X containing $E_1, \ldots, E_k$. □

Suppose that $E_1, \ldots, E_k$ are subspaces of X. Every element of the sum $E_1 + \cdots + E_k$ can be represented as

$$x_1 + \cdots + x_k$$

where $x_j \in E_j$ for $j = 1, \ldots, k$. If this representation is unique, we say that the sum $E_1 + \cdots + E_k$ is a **direct sum** and we denote it by $E_1 \oplus \cdots \oplus E_k$.

Example 13. Suppose that

$$E_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\} \quad \text{and} \quad E_2 = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}.$$

The sum $E_1 + E_2$ is a direct sum and is given by

$$E_1 \oplus E_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x, y \in \mathbb{F}\} = \mathbb{R}^3.$$

Example 14. For $j = 1, \ldots, n$, suppose that

$$E_j = \{(0, \ldots, 0, x, 0, \ldots, 0) \in \mathbb{F}^n \mid x \in \mathbb{R}\},$$

where the jth coordinate of $(0, \ldots, 0, x, 0, \ldots, 0)$ is equal to x. Then

$$E_1 \oplus \cdots \oplus E_n = \mathbb{F}^n.$$

Example 15. Consider

$$\begin{aligned} E_1 &= \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}, \\ E_2 &= \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}, \\ E_3 &= \{(0, y, y) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}. \end{aligned}$$

Then

$$E_1 + E_2 + E_3 = \mathbb{F}^3$$

because every vector (x, y, x) in $\mathbb{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0).$$

However, the sum $E_1 + E_2 + E_3$ is not a direct sum because vectors in it can be represented in different ways as a sum of vectors in E_1, E_2 and E_3. For example,

$$(1, 1, 1) = (1, 1, 0) + (0, 0, 1) + (0, 0, 0)$$

and

$$(1, 1, 1) = (1, 0, 0) + (0, 0, 0) + (0, 1, 1).$$

To decide whether a sum of subspaces is a direct sum, we need only to consider whether the vector zero can be uniquely written as a sum:

Theorem 4 (Condition for a direct sum)**.** *Suppose $E_1, \ldots, E_k$ are subspaces of X. Then $E_1 + \cdots + E_k$ is a direct sum if and only if the only way to write zero as a sum is $0 = 0 + \cdots + 0$.*

Proof. ($\Rightarrow$) Suppose $E_1 + \cdots + E_k$ is a direct sum. The vector zero can be written as $0 = 0 + \cdots + 0$. But since $E_1 + \cdots + E_k$ is a direct sum, this is the only way to express the vector zero, by definition.

($\Leftarrow$) Suppose that the only way to write zero is $0 = 0 + \cdots + 0$. Let $x \in E_1 + \cdots + E_k$. Then $x = x_1 + \cdots + x_k$ with $x_j \in E_j$ for $j = 1, \ldots, k$. Suppose that $x = y_1 + \cdots + y_k$ with $y_j \in E_j$ for $j = 1, \ldots, k$. Then $0 = (x_1 - y_1) + \cdots + (x_k - y_k)$ with $(x_j - y_j) \in E_j$ for $j = 1, \ldots, k$. Hence $x_j - y_j = 0$ for each j, that is, $x_j = y_j$. Thus the representation of x as a sum is unique. Therefore the sum is a direct sum. □

The next theorem gives a simple criteria for verifying whether a pair of subspaces form a direct sum:

Theorem 5 (Direct sum of two subspaces)**.** *Suppose that E_1 and E_2 are subspaces of X. Then $E_1 + E_2$ is a direct sum if and only if $E_1 \cap E_2 = \{0\}$.*

Proof. ($\Rightarrow$) Suppose that $E_1 + E_2$ is a direct sum. If $x \in E_1 \cap E_2$, we can write $0 = x + (-x)$ with $x \in E_1$ and $-x \in E_2$. By the unique representation of zero as a sum, it follows that $x = 0$. Thus $E_1 \cap E_2 = \{0\}$.

(⇐) Conversely, suppose that $E_1 \cap E_2 = \{0\}$ and $0 = x_1 + x_2$ with $x_1 \in E_1$ and $x_2 \in E_2$. This equality implies $x_1 = -x_2$. Hence $x_1 \in E_2$ because $-x_2 \in E_2$. Thus $x_1 \in E_1 \cap E_2$. Therefore $x_1 = 0$ and $x_2 = -x_1 = 0$. Thus the only way to write zero as a sum is $0 = 0 + 0$. Therefore $E_1 + E_2$ is a direct sum, by Theorem 4. □

The last theorem considers *only the case of two subspaces.* When asking whether the sum of more than two subspaces is a direct sum, it is not enough to verify that each pair of subspaces intersect only at zero. For instance, in Example 15, we have $E_1 \cap E_2 = \{0\}$, $E_2 \cap E_3 = \{0\}$ and $E_3 \cap E_1 = \{0\}$. However, the sum $E_1 + E_2 + E_2$ is not a direct sum.

Sums of subspaces are analogous to unions of subsets, and direct sums are analogous to unions of disjoint subsets.