Problem Set 1
Symmetry Properties of the Laplacian

1 Introduction

The purpose of this problem set is to prove the following theorem, which is a more general and precise version of a theorem that was mentioned in Lecture 1.

Theorem 1. *Suppose that* $L:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ *is a linear differential operator of order* m*, and suppose that* T *is a translation on* $\mathbb{R}^n$ *or a rotation on* $\mathbb{R}^n$ *(that is, an orthogonal transformation). Then* L *is invariant by* T*, that is,*

$$(Lf)\circ T = L(f\circ T) \quad \text{for all } f\in\mathcal{S}(\mathbb{R}^n),$$

if and only if

$$L=\sum_{j=0}^{\lfloor m/2\rfloor} c_j\Delta^j,$$

where $c_0, c_1, \dots, c_{\lfloor m/2\rfloor}$ *are constants. Here* $\lfloor m/2\rfloor$ *denotes the largest integer less than or equal to* $m/2$*.*

To prove this theorem, we will use the Fourier transform. The proof is divided in five questions in the end of this note (these form the actual problem set). In the remaining part of this introduction, we present some definitions and properties.

We work in $\mathbb{R}^n$. For $x, y\in\mathbb{R}^n$, we set

$$x\cdot y = x_1y_1+\cdots+x_ny_n \qquad \text{and} \qquad |x|=\sqrt{x\cdot x}.$$

For any non-negative integer k, we denote by $C^k(\mathbb{R}^n)$ the set of all the functions on $\mathbb{R}^n$ whose partial derivatives of order less than or equal to k exist and are continuous. We define $C^\infty(\mathbb{R}^n)=\cap_{k=1}^\infty C^k(\mathbb{R}^n)$. We denote by $C_c^\infty(\mathbb{R}^n)$ the set of all functions in $C^\infty(\mathbb{R}^n)$ whose support is compact and contained in $\mathbb{R}^n$. We denote by $C_0^\infty(\mathbb{R}^n)$ the set of all functions in $C^\infty(\mathbb{R}^n)$ which vanish at infinity.

Let K be a compact subset of $\mathbb{R}^n$. There exists a function $J_K\in C_c^\infty(\mathbb{R}^n)$ such that $0\le J_K(x)\le 1$ for all $x\in\mathbb{R}^n$ and $J_K(x)=1$ for all $x\in K$.

Multi-index notation. A *multi-index* is an n-tuple of non-negative integers. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, we define

$$|\alpha|_1 = \sum_{j=1}^{n} |\alpha_j|, \qquad \alpha! = \prod_{j=1}^{n} \alpha_j!, \qquad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

If $x \in \mathbb{R}^n$, we define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

The Schwartz space. The *Schwartz space* of functions on $\mathbb{R}^n$, denoted by $\mathcal{S}(\mathbb{R}^n)$, is defined by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{N,\alpha} < \infty \text{ for all } N, \alpha\},$$

where N is a non-negative integer, α is a multi-index, and

$$\|f\|_{N,\alpha} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|.$$

Thus a function in $\mathcal{S}(\mathbb{R}^n)$ and all its partial derivatives vanish at infinity faster than any power of $|x|$. For example, for any multi-index β, the function $x \mapsto x^\beta e^{-|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. We observe that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

Fourier transform. For $f \in L^1(\mathbb{R}^n)$, the *Fourier transform* of f, denoted by $\mathcal{F}f$ or $\hat{f}$, is the function on $\mathbb{R}^n$ defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx.$$

Clearly $\hat{f}(\xi)$ is well-defined for all $\xi \in \mathbb{R}^n$ and $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$. Thus $\hat{f}$ is a bounded function. In addition, $\hat{f}$ is continuous. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, the mapping $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is invertible.

The Fourier transform interacts in a simple way with compositions with translations, rotations, and differentiations:

Theorem 2. *Let $f \in L^1(\mathbb{R}^n)$. We have the following properties:*

(i) *If S is a translation on $\mathbb{R}^n$, that is, $S(x) = x + a$ for $a \in \mathbb{R}^n$, then $\mathcal{F}(f \circ S)(\xi) = e^{2\pi i a \cdot \xi} (\mathcal{F}f)(\xi)$.*

(ii) *If T is a rotation on $\mathbb{R}^n$, that is, T is an orthogonal transformation, then $\mathcal{F}(f \circ T) = (\mathcal{F}f) \circ T$.*

(iii) *If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for $|\alpha|_1 \leq k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for $|\alpha|_1 \leq k - 1$, then $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i \xi)^\alpha (\mathcal{F}f)(\xi)$.*

Differential operator. A *linear differential operator* on $\mathbb{R}^n$ of order m is an operator of the form

$$L = \sum_{|\alpha|_1 \leq m} a_\alpha \partial^\alpha,$$

where $a_\alpha \in C^\infty(\mathbb{R}^n)$ for every multi-index α. The functions a_α are called the *coefficients* of L. If the function a_α is constant for every α, the linear operator L is said to have *constant coefficients*.

For a constant $M > 0$, we define

$$\mathcal{P}_M(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \sup_{x\in\mathbb{R}^n} (1+|x|)^{-M} |\partial^\alpha f(x)| \le C_\alpha \text{ for all } \alpha\}.$$

Thus a function in $\mathcal{P}_M(\mathbb{R}^n)$ and all its partial derivatives grow at most as fast as $|x|^M$. If the coefficients of a linear operator L are elements of $\mathcal{P}_M(\mathbb{R}^n)$, then L maps the Schwartz space into itself: $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

Polynomials. A *polynomial* of degree m in n complex variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form $P(\xi) = \sum_{|\alpha|_1 \le m} b_\alpha \xi^\alpha$, where $b_\alpha \in \mathbb{C}$ for every multi-index α.

A polynomial is said to be *homogeneous of degree d* if $P(\lambda\xi) = \lambda^d P(\xi)$.

Every polynomial P of degree m can be written (in a unique way) as a sum of the form $P(\xi) = \sum_{j=0}^m P_j(\xi)$, where P_j is a homogeneous polynomial of degree j. The polynomials $P_1, P_2, \ldots, P_m$ are called the *homogeneous components of P*.

A function $F : \mathbb{R}^n \to \mathbb{R}$ (or in particular a polynomial) is said to be *rotation-invariant* if $F \circ T = F$ for all orthogonal transformations T on $\mathbb{R}^n$. If F is rotation-invariant, then $F(\xi) = f(|\xi|)$ for some function $f : \mathbb{R} \to \mathbb{R}$.

2 Problems

1. Let $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear differential operator of order m.

   (a) Show that L is invariant by translations if and only if L has constant coefficients. (Hint: Consider the action of L on the monomials x^α.)

   (b) Note that if L has constants coefficients, then L can be written as $L = \sum_{|\alpha|_1 \le m} b_\alpha D^\alpha$ with $b_\alpha = (2\pi i)^{|\alpha|_1} a_\alpha$ and $D^\alpha = (2\pi i)^{-|\alpha|_1} \partial^\alpha$. Hence we use the notation $L = P(D)$ where P is the polynomial $P(\xi) = \sum_{|\alpha|_1 \le m} b_\alpha \xi^\alpha$. (There is nothing to prove in this question.)

   (c) Show that if L has constant coefficients, then L is invariant by rotations if and only if P is invariant by rotations.

   (d) Prove that P is invariant by rotations if and only if each of its homogeneous components P_j is invariant by rotations. (Hint: Use induction on j and the formula $P_j(\xi) = \lim_{r\to 0} r^{-j} \sum_{k=j}^m P_k(r\xi)$.)

   (e) Show that if P_j is invariant by rotations, then $P_j(\xi) = b_j |\xi|^j$ for some constant b_j with $b_j = 0$ if j is odd. Conclude that

   $$P(\xi) = \sum_{j=0}^{\lfloor m/2 \rfloor} c_j (-4\pi^2 |\xi|^2)^j$$

   with $c_j = (-4\pi^2)^{-j} b_{2j}$ and consequently that $L = \sum_{j=0}^{\lfloor m/2 \rfloor} c_j \Delta^j$.