Problem Set 2
Operators on Hilbert Spaces

1 Problems

1. Let $(a_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Consider the operator

$$M_a : \begin{cases} l^2 \to l^2 \\ (x_1, x_2, x_3, \dots) \mapsto (a_1x_1, a_2x_2, a_3x_3, \dots). \end{cases}$$

   (a) Prove that M_a is bounded if and only if $(a_j)_{j\in\mathbb{N}}$ is bounded.

   (b) Prove that if M_a is bounded then $\|M_a\| = \sup_{j\in\mathbb{N}} |a_j|$.

2. Is the operator

$$M : \begin{cases} l^2 \to l^2 \\ (x_j)_{j\in\mathbb{N}} \mapsto (\frac{1}{j}x_j)_{j\in\mathbb{N}} \end{cases}$$

   bijective? Give an example to support your answer.

3. For $k = 1, \dots, 5$, consider the operators

$$T_k : \begin{cases} D(T_k) \to L^2([a,b]) \\ f \mapsto if' \end{cases}$$

   with

$$\begin{aligned} D(T_1) &= H^1((a,b)), \\ D(T_2) &= \{f \in H^1((a,b)) \mid f(a) = 0\}, \\ D(T_3) &= \{f \in H^1((a,b)) \mid f(b) = 0\}, \\ D(T_4) &= \{f \in H^1((a,b)) \mid f(a) = f(b)\}, \\ D(T_5) &= H^1_0((a,b)) = \{f \in H^1((a,b)) \mid f(a) = f(b) = 0\}. \end{aligned}$$

   In Lecture 2, we mentioned that the operators $T_1, \dots, T_5$ are closed with dense domains.

   (a) Prove that T_1 is closed with dense domain.

   (b) Prove that T_2 is closed with dense domain.

(c) Conclude that the strategy used to prove (b) may be used to prove that T_3, T_4 and T_5 are closed with dense domains.

To solve this problem, you may use basic theorems of analysis and theorems mentioned in class. In particular, the following theorem may be useful:

Theorem 1 (Sobolev embedding)**.** *The space $H^1((a,b))$ is a subset of $C([a,b])$. Furthermore, there exists a constant $C > 0$ such that*

$$\|f\|_u \leq C\|f\|_{H^1((a,b))} \quad \textit{for all } f \in H^1((a,b)).$$

Here $\|f\|_u = \sup_{x\in[a,b]} |f(x)|$.

An element of $H^1((a,b))$ is an equivalence class of functions that are equal almost everywhere, and the Sobolev embedding theorem states that each such equivalence class contains a continuous function on $[a,b]$. Equivalently, the Sobolev embedding theorem states that there exists a continuous mapping (or embedding)

$$J : H^1((a,b)) \to (C([a,b]), \|\cdot\|_u)$$

that identifies a function f regarded as an element of $H^1((a,b))$ with the same function f regarded as an element of $C([a,b])$.

4. Consider the operators $T_1, \ldots, T_5$ in Problem 3. In Lecture 2, we mentioned that $T_1^* = T_5$, $T_2^* = T_3$, $T_3^* = T_2$, $T_4^* = T_4$, and $T_5^* = T_1$. Prove that $T_1^* = T_5$.