Problem Set 3 – Solutions

Problem 1. (a) ($\Leftarrow$) Suppose that $\lim_{j\to\infty} a_j = 0$. For $n \in \mathbb{N}$, we define $M_a^{(n)}$ by

$$M_a^{(n)}x = (a_1x_1, a_2x_2, \dots, a_nx_n, 0, 0, \dots).$$

We obtain that

$$(M_a - M_a^{(n)})x = (0, 0, \dots, 0, a_{n+1}x_{n+1}, a_{n+2}x_{n+2}, \dots).$$

Furthermore

$$\|M_a - M_a^{(n)}\| = \sup_{j\geq n+1} |a_j| \longrightarrow 0 \text{ as } j \to \infty.$$

Since each $M_a^{(n)}$ has finite rank, each $M_a^{(n)}$ is compact. Since $(M_a^{(n)})_{n\in\mathbb{N}}$ converges to $M_a^{(n)}$ with respect to the operator norm, it follows by a theorem (which we saw in class) that M_a is compact.

($\Rightarrow$) We prove by contraposition. Suppose that $\lim_{j\to\infty} a_j \neq 0$. Then there exists a subsequence $(a_{j_k})_k$ such that $|a_{j_k}| \geq \varepsilon$ for $\varepsilon > 0$. Consider the sequence of the vectors e_{j_k} of the standard basis. For any $k, m \in \mathbb{N}$, we have $\|e_{j_k}\| = 1$ and

$$\|M_ae_{j_m} - M_ae_{j_k}\|^2 = \|a_{j_m}e_{j_m} - a_{j_k}e_{j_k}\|^2 = |a_{j_m}|^2 + |a_{j_k}|^2 \geq 2\varepsilon^2 > 0.$$

Thus the sequence $M_ae_{j_k}$ does not contain a convergent subsequence. Hence M_a is not compact.

(b) ($\Rightarrow$) Suppose that M_a is bijective and has a bounded inverse. For $j \in \mathbb{N}$, let e_j be the jth vector from the standard basis of l^2. Since $M_ae_j \neq 0$ for all $j \in \mathbb{N}$, from the equality $M_ae_j = a_je_j$ we conclude that $a_j \neq 0$. Furthermore $M_a^{-1}e_j = a_j^{-1}e_j$. Hence $\|M_a^{-1}\| \geq a_j^{-1}$, that is, $a_j \geq \|M_a^{-1}\|^{-1}$. This implies $\inf_{j\in\mathbb{N}} |a_j| > 0$.

($\Leftarrow$) Conversely, suppose that $\inf_{j\in\mathbb{N}} |a_j| > 0$. Then $\sup_{j\in\mathbb{N}} |a_j|^{-1} \leq \infty$. Define M_a^{-1} by $M_a^{-1}x = (a_1^{-1}x_1, a_2^{-1}x_2, \dots)$. By Problem 1 of Problem Set 2, this defines a bounded operator on l^2. Furthermore, it is easy to check that $M_aM_a^{-1}x = x$ and $M_a^{-1}M_ax = x$ for all $x \in l^2$. Thus M_a is bijective and has a bounded inverse.

(c) We have

$$(M_a - \lambda I)x = \big((a_1 - \lambda)x_1, (a_2 - \lambda)x_2, \dots\big).$$

By Part (b), this operator is invertible and has a bounded inverse if and only if $\inf_{j\in\mathbb{N}} |a_j - \lambda| > 0$, that is, $\lambda \notin \overline{\{a_j \mid j \in \mathbb{N}\}}$. Thus $\sigma(M_a) = \overline{\{a_j \mid j \in \mathbb{N}\}}$. Finally, it is clear that $\{a_j \mid j \in \mathbb{N}\} \subseteq \sigma(T)$. We want to show that $\{a_j \mid j \in \mathbb{N}\} = \sigma_p(T)$. Suppose that $\alpha \in \overline{\{a_j \mid j \in \mathbb{N}\}}$ and $\alpha \neq a_j$ for all $j \in \mathbb{N}$. Then

$$(M_a - \alpha I)x = ((a_1 - \alpha)x_1, (a_2 - \alpha)x_2, \dots) = (0, 0, \dots)$$

implies $x = 0$. Thus α is not an eigenvalue. Therefore $\sigma_p(T) = \{a_j \mid j \in \mathbb{N}\}$.

Problem 2. (a) We first observe that

$$\|Tf\|^2 = \int_0^2 \left| \int_0^1 k(x,y)f(y)\,dy \right|^2 dx \leq \int_0^1 \left(\int_0^1 |k(x,y)f(y)|\,dy \right)^2 dx.$$

On the other hand, by Hölder's inequality, we have

$$\int_0^1 |k(x,y)f(y)|\,dy \leq \left(\int_0^1 |k(x,y)|^2\,dy \right)^{1/2} \left(\int_0^1 |f(y)|\,dy \right)^{1/2}$$

for almost all $x \in [0,1]$. Hence

$$\|Tf\|^2 \leq \int_0^1 |f(y)|^2\,dy \int_0^1 \int_0^1 |k(x,y)|^2\,dxdy = \|f\|^2 \|k\|^2_{L^2([0,1]\times[0,1])}.$$

Thus $\|T\| \leq \|k\|_{L^2([0,1]\times[0,1])}$.

(b) Recall that if $\{u_j \mid j \in \mathbb{N}\}$ is an orthonormal basis for $L^2([0,1])$, then $\{(x,y) \mapsto u_j(x)u_k(y) \mid j,k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([0,1]\times[0,1])$. Using this fact, we may write

$$k(x,y) = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} c_{jk} u_j(x) u_k(y)$$

where the series converges with respect to the L^2-norm. For $n \in \mathbb{N}$, define

$$k_n(x,y) = \sum_{j=1}^n \sum_{k=1}^n c_{jk} u_j(x) u_k(y).$$

Then $\|k_n - k\|_{L^2([0,1]\times[0,1])} \longrightarrow 0$ as $n \to \infty$. Let T_n be the operator on $L^2([0,1])$ defined by

$$(T_n f)(x) = \int_0^1 k_n(x,y) f(y)\,dxdy.$$

The rank of the operator T_n is less than n. Thus T_n is a compact operator. By Part (a), it follows that $\|T_n - T\| \longrightarrow 0$ as $n \to \infty$. Hence T is a compact operator.

Problem 3. We observe that T is a compact operator. Thus $0 \in \sigma(T)$ and any $\lambda \in \sigma(T)$ with $\lambda \neq 0$ is an eigenvalue. Suppose that $\lambda \in \sigma(T)$ and $\lambda \neq 0$. For all $x \in [0,1]$, on the one hand we have

$$(Tf)(x) = \lambda f(x).$$

On the other hand

$$\begin{aligned}(Tf)(x) &= \int_0^1 k(x,y) f(y)\, dy \\ &= \int_0^x k(x,y) f(y)\, dy + \int_x^y k(x,y) f(y)\, dy \\ &= \int_0^x f(y)\, dy.\end{aligned}$$

Hence

$$\int_0^x f(y)\, dy = \lambda f(x).$$

This implies that $f \in H^1((0,1))$. By the Sobolev embedding Theorem (in Problem Set 2), we conclude that f is continuous. The above relation implies that f is continuously differentiable, $\lambda f' = f$ and $f(0) = 0$. By solving this equation we find that $f(x) = 0$ for all $x \in [0,1]$, that is, $f = 0$. In this case f is not an eigenvector and consequently $\lambda \notin \sigma_p(T)$. Therefore $\sigma(T) = \{0\}$.