Problem Set 4 – Solutions

Problem 1. (a) Let $x \in S^n$. We have $x_1^2 + \cdots x_{n+1}^2 = 1$. Hence $x_j \neq 0$ for some $j \in \{1, \dots, n+1\}$. Thus either $x_j > 0$ or $x_j < 0$ for some $j \in \{1, \dots, n+1\}$. This implies that either $x \in \mathcal{U}_j$ or $x \in \mathcal{V}_j$ for some $j \in \{1, \dots, n+1\}$. That is $x \in \mathcal{U}_j \cup \mathcal{V}_j$ for some $j \in \{1, \dots, n+1\}$. Hence $x \in \bigcup_{j=1}^{n+1} \mathcal{U}_j \cup \mathcal{V}_j$. Thus $S^n \subset \bigcup_{j=1}^{n} \mathcal{U}_j \cup \mathcal{V}_j$. Clearly $\bigcup_{j=1}^{n} \mathcal{U}_j \cup \mathcal{V}_j \subset S^n$. Therefore $S^n = \bigcup_{j=1}^{n} \mathcal{U}_j \cup \mathcal{V}_j$. We observe that φ_j and ψ_j are invertible on $\varphi_j(\mathcal{U}_j)$ and $\psi_j(\mathcal{V}_j)$ with

$$\begin{aligned}
&\varphi_j^{-1}(y_1, \dots, y_n) \\
&= (y_1, \dots, y_{j-1}, \sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_j, \dots, y_n), \\
&\psi_j^{-1}(y_1, \dots, y_n) \\
&= (y_1, \dots, y_{j-1}, -\sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_j, \dots, y_n).
\end{aligned}$$

It follows that φ_j and ψ_j are homeomorphisms. We observe that $\mathcal{U}_j$ and $\mathcal{V}_j$ are connected. Thus $(\mathcal{U}_j, \varphi_j)$ and $(\mathcal{V}_j, \psi_j)$ are charts. To prove that these charts are compatible we need to verify that $\varphi_j \circ \varphi_k^{-1}$, $\varphi_j \circ \psi_k^{-1}$, $\psi_j \circ \psi_k^{-1}$ and $\psi_j \circ \varphi_k^{-1}$ are smooth. We consider $\varphi_j \circ \varphi_k^{-1}$ for $j < k$. The other cases are similar. We have

$$\begin{aligned}
&\varphi_j \circ \varphi_k^{-1}(y_1, \dots, y_n) \\
&= \varphi_j(y_1, \dots, y_{k-1}, \sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_k, \dots, y_n) \\
&= (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{k-1}, \sqrt{1 - (y_1^2 + \cdots + y_n^2)}, y_k, \dots, y_n).
\end{aligned}$$

It is simple to verify that this mapping is smooth. Combining all this we conclude that $\mathcal{A}_1$ is an atlas for S^n.

(b) Let $x \in S^n$. If $x_{n+1} = 1$, then $x \in \mathcal{V}$, else $x \in \mathcal{U}$. Thus $x \in \mathcal{U} \cup \mathcal{V}$. Hence $S^n \subset \mathcal{U} \cup \mathcal{V}$. Clearly $\mathcal{U} \cup \mathcal{V} \subset S^n$. Therefore $S^n = \mathcal{U} \cup \mathcal{V}$.

We now prove that φ and ψ are homeomorphisms. We give the details of the proof only for φ. The proof for ψ is similar. Let $y = \varphi(x)$. We have

$$y_j = \frac{2x_j}{1 - x_{n+1}}$$

for $j = 1, \ldots, n$. We calculate

$$\begin{aligned}|y|^2 &= \frac{4}{(1-x_{n+1})^2}(|x|^2 - x_{n+1}^2)\\ &= \frac{4}{(1-x_{n+1})^2}(1 - x_{n+1}^2)\\ &= \frac{4}{(1-x_{n+1})^2}((1-x_{n+1})^2 + 2x_{n+1}(1-x_{n+1}))\\ &= 4 + \frac{8x_{n+1}}{1-x_{n+1}}.\end{aligned}$$

Solving this equation for x_{n+1}, we obtain

$$x_{n+1} = \frac{|y|^2 - 4}{|y|^2 + 4}.$$

Hence

$$x_j = \frac{4y_j}{|y|^2 + 4}$$

for $j = 1, \ldots, n$. Thus, for $y \in \varphi(\mathcal{U})$, we have

$$\varphi^{-1}(y) = \left(\frac{4y_1}{|y|^2+4}, \ldots, \frac{4y_n}{|y|^2+4}, \frac{|y|^2-4}{|y|^2+4}\right).$$

In fact, it follows that $\varphi \circ \varphi^{-1}(y) = y$ and $\varphi^{-1} \circ \varphi(x) = x$. It is simple to verify that φ is differentiable. Thus it follows that φ is a homeomorphism. Similarly we find that

$$\psi^{-1}(y) = \left(\frac{4y_1}{|y|^2+4}, \ldots, \frac{4y_n}{|y|^2+4}, \frac{4-|y|^2}{|y|^2+4}\right)$$

for $y \in \psi(\mathcal{V})$. It follows that ψ is a homeomorphism. We observe that $\mathcal{U}$ and $\mathcal{V}$ are connected. Thus $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ are charts. To prove that these charts are compatible we need to verify that $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are smooth. We consider $\varphi \circ \psi^{-1}$. The other case is similar. We have

$$\begin{aligned}(\varphi \circ \psi^{-1})_j(y) &= \varphi_j\left(\frac{4y_1}{|y|^2+4}, \ldots, \frac{4y_n}{|y|^2+4}, \frac{4-|y|^2}{|y|^2+4}\right)\\ &= 2\frac{4y_j}{|y|^2+4}\left(1 - \frac{4-|y|^2}{|y|^2+4}\right)^{-1}\\ &= \frac{4y_j}{|y|^2}.\end{aligned}$$

We observe that $|y| \neq 0$. It is simple to verify that this mapping is smooth. We conclude that $\mathcal{A}_2$ is an atlas.

(c) By proceeding similarly as in Part (b), we obtain

$$(\psi \circ \varphi^{-1})(y) = \frac{4}{|y|^2} y.$$

(d) Consider the chart $(\mathcal{U}_j, \varphi_j)$. For $k = 1, \ldots, n$, the vector

$$\frac{\partial \varphi_j^{-1}}{\partial y_k}(y) \in \mathbb{R}^{n+1}$$

corresponds to the tangent vector

$$\frac{\partial}{\partial y_k} \in T_{\varphi_j(y)} S^n.$$

We calculate

$$\begin{aligned} \frac{\partial \varphi_j^{-1}}{\partial y_k}(y) &= \left(\frac{\partial y_1}{\partial y_k}, \ldots, \frac{\partial y_{j-1}}{\partial y_k}, \frac{\partial}{\partial y_k} \sqrt{1 - |y|^2}, \frac{\partial y_j}{\partial x_k}, \ldots, \frac{\partial y_n}{\partial y_k} \right) \\ &= \left(\delta_{1,k}, \ldots, \delta_{j-1,k}, \frac{-y_k}{\sqrt{1 - |y|^2}}, \delta_{j,k}, \ldots, \delta_{n,k} \right) \end{aligned}$$

Similarly we obtain

$$\frac{\partial \psi^{-1}}{\partial y_k}(y) = \left(\delta_{1,k}, \ldots, \delta_{j-1,k}, \frac{y_k}{\sqrt{1 - |y|^2}}, \delta_{j,k}, \ldots, \delta_{n,k} \right).$$

(e) Working in S^1 with the atlas $\mathcal{A}_1$, we have four charts:

$$\mathcal{A}_1 = \left\{ (\varphi_1, \mathcal{U}_1), (\varphi_2, \mathcal{U}_2), (\psi_1, \mathcal{V}_1), (\psi_2, \mathcal{V}_2) \right\}.$$

Consider the chart $(\varphi_1, \mathcal{A}_1)$. We have

$$\varphi_1(x_1, x_2) = x_2$$

and

$$\varphi_1^{-1}(y_1) = (\sqrt{1 - y_1^2}, y_1).$$

Hence $\frac{\partial}{\partial y_1}$ corresponds to

$$\frac{\partial \varphi_1^{-1}}{\partial y_1}(y_1) = \left(\frac{-y_1}{\sqrt{1 - y_1^2}}, 1 \right).$$

(Similarly we calculate $\partial/\partial y_1$ on the other charts.) Thus

$$g_{11}(y_1) = \left\langle \frac{\varphi_1^{-1}}{\partial y_1}(y_1), \frac{\varphi_1^{-1}}{\partial y_1}(y_1) \right\rangle = \frac{1}{1 - y_1^2}$$

and

$$g^{11}(y_1) = g_{11}^{-1}(y_1) = 1 - y_1^2.$$

(Similarly we obtain the same expressions for g_{11} and g^{11} on the other charts.) Furthermore

$$\mathrm{grad}(f) = g^{11} \frac{\partial f}{\partial y_1} \frac{\partial}{\partial y_1} = (1 - y_1^2) \frac{\partial f}{\partial y_1} \frac{\partial}{\partial y_1}$$

and

$$\mathrm{div}(X) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_1} (\sqrt{g} X^1)$$

where

$$X = X^1 \frac{\partial}{\partial y_1}.$$

Hence

$$\begin{aligned}\Delta_g(f) &= \mathrm{div}(\mathrm{grad}(f)) \\ &= \sqrt{1 - y_1^2} \frac{\partial}{\partial y_1} \left(\sqrt{1 - y_1^2} \frac{\partial f}{\partial y_1} \right) = (1 - y_1^2) \frac{\partial^2 f}{\partial y_1^2} - y_1 \frac{\partial f}{\partial y_1}.\end{aligned}$$

Therefore

$$\Delta_g = (1 - y_1^2) \frac{\partial^2}{\partial y_1^2} - y_1 \frac{\partial}{\partial y_1}.$$

(Similarly we obtain the *same* expression on the other charts.)

Problem 2. Let $(x_1, \ldots, x_m)$ be a coordinate system for M, and let $(y_1, \ldots, y_n)$ be a coordinate system for N. We first calculate

$$\begin{aligned}\left(\Phi_* \frac{\partial}{\partial x_j} \right) f &= \frac{\partial}{\partial x_j} (\Phi^* f) = \frac{\partial}{\partial x_j} (f \circ \Phi) = \sum_{k=1}^n \frac{\partial f}{\partial y_k} \frac{\partial \Phi_k}{\partial x_j} \\ &= \left(\sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_j} \frac{\partial}{\partial y_k} \right) f.\end{aligned}$$

Now, since $\Phi^* g_N = g_M$, we have

$$\begin{aligned}g_{ij}^M &= g_M \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \Phi^* g_N \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \\ &= g_N \left(\Phi_* \frac{\partial}{\partial x_i}, \Phi_* \frac{\partial}{\partial x_j} \right) = g_N \left(\sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_i} \frac{\partial}{\partial y_k}, \sum_{l=1}^n \frac{\partial \Phi_l}{\partial x_j} \frac{\partial}{\partial y_l} \right) \\ &= \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_i} \sum_{l=1}^n g^N \left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right) \frac{\partial \Phi_l}{\partial x_j} \\ &= [D\Phi^T G_N D\Phi]_{ij}.\end{aligned}$$

Hence

$$G_M = D\Phi^T G_N D\Phi.$$

This implies $\det(D\Phi) \neq 0$. Thus $D\Phi$ is a local isomorphism. Hence, by the inverse function theorem, the mapping Φ is a local diffeomorphism. In addition, the equality

$$D\Phi(\Phi^{-1}(q))^{-1} D\Phi(\Phi^{-1}(q) = I$$

implies

$$D\Phi^{-1}(q) D\Phi(\Phi^{-1}(q)) = I$$

because $D\Phi(\Phi^{-1}(q))^{-1} = D\Phi^{-1}(q)$.

(a) Using the above observations, we obtain

$$\begin{aligned}
\Phi_*(\mathrm{grad}_{g_M} \Phi^* f) &= \Phi_*(\mathrm{grad}_{g_M} f \circ \Phi) \\
&= \Phi_* \left(\sum_{j=1}^{m} (\mathrm{grad}_{g_M} f \circ \Phi)^j \frac{\partial}{\partial x_j} \right) \\
&= \Phi_* \left(\sum_{j=1}^{m} \sum_{k=1}^{m} g_M^{jk} \frac{\partial}{\partial x_k} (f \circ \Phi) \frac{\partial}{\partial x_j} \right) \\
&= \sum_{j=1}^{m} \sum_{k=1}^{m} g_M^{jk} \sum_{l=1}^{n} \frac{\partial f}{\partial y_l} \frac{\partial \Phi_l}{\partial x_k} \Phi_* \frac{\partial}{\partial x_j} \\
&= \sum_{q=1}^{n} \sum_{l=1}^{n} \left[\sum_{j=1}^{m} \frac{\partial \Phi_q}{\partial x_j} \sum_{k=1}^{m} g_M^{jk} \frac{\partial \Phi_l}{\partial x_j} \right] \frac{\partial f}{\partial y_l} \frac{\partial}{\partial y_q} \\
&= \sum_{q=1}^{n} \sum_{l=1}^{n} [D\Phi G_M^{-1} D\Phi^T]_{ql} \frac{\partial f}{\partial y_l} \frac{\partial}{\partial y_q} \\
&= \sum_{q=1}^{n} \sum_{l=1}^{n} g_N^{ql} \frac{\partial f}{\partial y_l} \frac{\partial}{\partial y_q} \\
&= \sum_{q=1}^{n} (\mathrm{grad}_{g_N} f)^q \frac{\partial}{\partial y_q} \\
&= \mathrm{grad}_{g_n} f.
\end{aligned}$$

(b) We will make several observations and combine them in the end to obtain the answer. First

$$\Phi^*(\mathrm{div}_{g_N} \Phi_* X) = (\mathrm{div}_{g_N} \Phi_* X) \circ \Phi.$$

Second

$$\mathrm{div}_{g_N} \Phi_* X = \frac{1}{\sqrt{\det G_N}} \sum_{j=1}^{n} \frac{\partial}{\partial y_j} (\sqrt{\det G_N} (\Phi_* X)^j).$$

Third, the equality

$$G_M(p) = (D\Phi)(p)^T G_N(\Phi(p))(D\Phi)(p)$$

implies

$$\sqrt{\det G_N(q)} = \frac{\sqrt{\det G_M(\Phi^{-1}(p))}}{|\det D\Phi(\Phi^{-1}(q))|}.$$

Fourth, the equality

$$(\Phi_* X)_1 = D\Phi(\Phi^{-1}(q))X_{\Phi^{-1}(q)}$$

implies

$$(\Phi_* X)^j(q) = \sum_{l=1}^{m} \frac{\partial \Phi_j}{\partial x_l}(\Phi^{-1}(q))X^l(\Phi^{-1}(q)).$$

Fifth

$$\sum_{j=1}^{n} \frac{\partial \Phi_j}{\partial x_l}(\Phi^{-1}(q))\frac{\partial}{\partial y_j}(f \circ \Phi^{-1})(q) = \frac{\partial f}{\partial x_l}(\Phi^{-1}(q)).$$

Sixth

$$\frac{d}{ds}\det T(s) = (\det T(s))\,\mathrm{Tr}\left(T^{-1}(s)\frac{d}{ds}T(s)\right).$$

Seventh

$$\sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\frac{1}{|D\Phi(\Phi^{-1}(q))|}\frac{\partial \Phi_j}{\partial x_l}(\Phi^{-1}(q))\right) = 0.$$

To prove this, observe that

$$\frac{\partial \Phi_k}{\partial x_k \partial x_l} = \frac{\partial \Phi_k}{\partial x_l \partial x_k}.$$

Combining all this, we obtain

$$\begin{aligned}
&(\mathrm{div}_{g_n}\Phi_* X)(q) \\
&= \frac{|\det D\Phi(\Phi^{-1}(q))|}{\sqrt{G_N(\Phi^{-1}(q))}} \\
&\quad \times \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\frac{\sqrt{\det G_M(\Phi^{-1}(q))}}{|\det D\Phi(\Phi^{-1})|}\sum_{l=1}^{m}\frac{\partial \Phi_j}{\partial x_l}(\Phi^{-1}(q))X^l(\Phi^{-1}(q))\right) \\
&= (\mathrm{div}_{g_M} X)(\Phi^{-1}(q)) + 0.
\end{aligned}$$

This implies the desired result.

(c) Using Parts (a) and (b), we obtain

$$\begin{aligned}
\Delta_{g_M}(\Phi^* f) &= \mathrm{div}_{g_M}(\mathrm{grad}_{g_M}\Phi^* f) \\
&= \Phi^*(\mathrm{div}_{g_N}\Phi_* \mathrm{grad}_{g_M}\Phi^* f) \\
&= \Phi^*(\mathrm{div}_{g_N}\mathrm{grad}_{g_N} f) \\
&= \Phi^*(\Delta_{g_N} f).
\end{aligned}$$

(d) Using that $\Phi^* g = g$ and the change of variable formula for integrals, we obtain

$$
\begin{aligned}
\langle \Phi^* f, \Phi^* g \rangle &= \int_M \Phi^* f \Phi^* g \, dV_g \\
&= \int_M \Phi^* f \Phi^* g \, dV_{\Phi^* g} \\
&= \int_M \Phi^* (f g \, dV_g) \\
&= \int_{\Phi(M)} f g \, dV_g \\
&= \int_M f g \, dV_g \\
&= \langle f, g \rangle
\end{aligned}
$$