Problem Set 4
Manifolds and the Laplacian

1 Problems

1. For $n \in \{1, 2, 3, \dots\}$, consider the set

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

and for $j \in \{1, \dots, n+1\}$, define

$$\mathcal{U}_j = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_j > 0\},$$
$$\varphi_j(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1})$$

and

$$\mathcal{V}_j = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_j < 0\},$$
$$\psi_j(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1})$$

and

$$\mathcal{A}_1 = \{(\mathcal{U}_j, \varphi_j), (\mathcal{V}_j, \psi_j) \mid j \in \{1, \dots, n+1\}\}.$$

The set S^n with the atlas $\mathcal{A}_1$ is a smooth manifold, called the n-sphere. The coordinate mappings φ_j and ψ_j are projections onto $\mathbb{R}^n$, viewed as the $\{x_j = 0\}$-hyperplane of $\mathbb{R}^{n+1}$. Another possible atlas for S^n, compatible with $\mathcal{A}_1$, is

$$\mathcal{A}_2 = \{(\mathcal{U}, \varphi), (\mathcal{V}, \psi)\}$$

where

$$\mathcal{U} = S^n \setminus \{(0, \dots, 0, 1)\},$$
$$\varphi(x_1, \dots, x_{n+1}) = \left(\frac{2x_1}{1 - x_{n+1}}, \dots, \frac{2x_n}{1 - x_{n+1}}\right)$$

and

$$\mathcal{V} = S^n \setminus \{(0, \dots, 0, -1)\},$$
$$\psi(x_1, \dots, x_{n+1}) = \left(\frac{2x_1}{1 + x_{n+1}}, \dots, \frac{2x_n}{1 + x_{n+1}}\right).$$

The coordinate mappings φ and ψ are the stereographic projections from the north and south poles, respectively. The range of both φ and ψ are $\mathbb{R}^n$. Thus we can think of S^n as $\mathbb{R}^n$ with an additional "point at infinity".

(a) Prove that $\mathcal{A}_1$ is an atlas for S^n.

(b) Prove that $\mathcal{A}_2$ is an atlas for S^n.

(c) Consider S^n with the atlas $\mathcal{A}_2$. Derive the expression for the transition function $\psi \circ \varphi^{-1}$.

(d) Consider S^n with the atlas $\mathcal{A}_1$. Working in each chart, find the expressions for the vectors in $\mathbb{R}^{n+1}$ corresponding to the tangent vectors $\partial/\partial x_1|_p, \ldots, \partial/\partial x_n|_p$ in T_pS^n.

(e) Consider S^1 with the atlas $\mathcal{A}_1$ and with the metric inherited from $\mathbb{R}^2$. Find local coordinate expressions for $g_{ij}(p)$, $g^{ij}(p)$ and Δ_g.

2. Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth mapping. We associate to Φ the mappings $\Phi^* : C(N) \to C(M)$ and $\Phi_* : TM \to TN$ in the standard way:

$$\Phi^* f = f \circ \Phi$$

and

$$(\Phi_* X)f = X(\Phi^* f).$$

Similarly, to any Riemannian metric g_N on N we associate a 2-form $\Phi^* g_N$ on M defined by

$$(\Phi^* g_N)(X, Y) = g_N(\Phi_* X, \Phi_* Y).$$

If g_M is a Riemannian metric on M, we say that Φ is a local isometry of (M, g_M) onto (N, g_N) if $\Phi^* g_N = g_M$. We say that Φ is an isometry if Φ is a local isometry and a diffeomorphism.

Following the above notation, suppose that Φ is a local isometry. Prove the following statements:

(a) For any $f \in C^1(N)$, we have

$$\Phi_*(\mathrm{grad}_{g_M} \Phi^* f) = \mathrm{grad}_{g_N} f.$$

(b) For any vector field X on M for which $\Phi_* X$ is a well-defined global vector field on N, we have

$$\Phi^*(\mathrm{div}_{g_N} \Phi_* X) = \mathrm{div}_{g_M} X.$$

(c) For any $f \in C^2(N)$, we have

$$\Delta_{g_M}(\Phi^* f) = \Phi^*(\Delta_{g_N} f).$$

(d) If $N = M$ and $g_N = g_M$, then Φ is a local isometry of (M, g_M) onto itself and Φ^* acts as an orthogonal transformation on $L^2(M)$.